A Romp Tour of Trigonometry (Sample)

Charles Rambo

Preface

Hello everybody! Thank you for checking out my preview of *A Romp Tour of Trigonometry*. I hope you find it to be a helpful resource.

Please email me at `charles.tutoring@gmail.com` if you notice any mistakes. The pages in this sample were taken from the same source as my book, so some references no longer work because the corresponding content isn't available in the sample. You'll have to buy the book to fix that.

If you are interested in buying a full copy of *A Romp Tour of Trigonometry*, please go to its `Amazon page`. *A Romp Tour of Trigonometry* costs $17.50 and contains 504 pages of material. It was designed for trigonometry students who would like to go into STEM majors as well as more advanced students that need a trigonometry reference.

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Contents

Chapter 4

Right Triangle Trigonometry

In this chapter, we will study the trigonometry of right triangles. For pedagogic reasons, a definition of the trigonometric functions will be delayed until Chapter 5. We assume knowledge of Chapters **??**, **??**, and **??**. Readers unfamiliar with square roots may find Appendix **??** helpful. Because trigonometric functions require either a calculator or a table to be evaluated at most angle measures, it is assumed that the reader has access to a scientific calculator and knowledge of its basic functionality.

4.1 Introduction to Trigonometric Functions

The three trigonometric functions that we will study in this chapter are sine, cosine, and tangent. Their values at $\angle A$ are denoted by

$$\sin A, \quad \cos A, \quad \text{and} \quad \tan A,$$

respectively.

For the time being, we will rely on our calculators, instead of a definition, for the correspondence between angle measures and the outputs of each trigonometric function.

Example 4.1 Use a calculate to find each of the following. Round to three decimal places.

(a) $\sin 72°$

(b) $\cos 22°$

(c) $\tan 27°$

(d) $3 \sin 10°$

Solution Make sure your calculate is set to degree mode when you evaluate.

(a) $\sin 72° \approx 0.951$

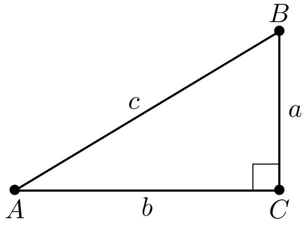
(b) $\cos 22° \approx 0.927$

(c) $\tan 27° \approx 0.510$

(d) $3 \sin 10° \approx 0.521$

■

Our next theorem, Theorem 4.1, is very powerful because it relates the values obtained from evaluation of trigonometric functions to ratios of a right triangle's side lengths. As a result of Theorem 4.1, when given a side length and an angle measure of a right triangle, we can find the other two side lengths using only a bit of algebra.



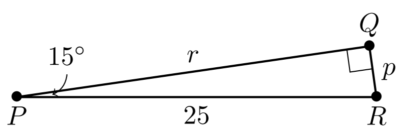
Theorem 4.1 *For any $\triangle ABC$ where $\angle C$ is right, the following trigonometric ratios hold.*

$$\begin{aligned} \sin A &= \frac{a}{c} \quad \textit{and} \quad & \sin B &= \frac{b}{c} \\ \cos A &= \frac{b}{c} & \cos B &= \frac{a}{c} \\ \tan A &= \frac{a}{b} & \tan B &= \frac{b}{a}. \end{aligned}$$

Many students remember these relationships by means of a mnemonic device such as the following.

SohCahToa

Sine is **o**pposite over **h**ypotenuse.
Cosine is **a**djacent over **h**ypotenuse.
Tangent is **o**pposite over **a**djacent.



Example 4.2 Find the lengths of the remaining sides of the triangle.

Solution Let us start by finding p. We know the length of the hypotenuse and we want to find the length of the leg opposite $\angle P$. Theorem 4.1 tells us sine relates these sides. In particular, it says

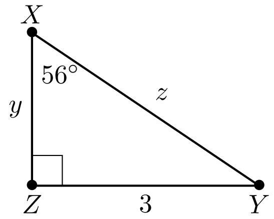
$$\sin 15^\circ = \frac{p}{25} \quad \text{implies} \quad p = 25 \sin 15^\circ \approx 6.470.$$

Let us find r. We know the hypotenuse and we want to find the length of the leg adjacent to $\angle P$. According to Theorem 4.1, cosine relates these sides. Specifically,

$$\cos 15^\circ = \frac{r}{25} \quad \text{implies} \quad r = 25 \cos 15^\circ \approx 24.148.$$

■

Notice that we could have used tangent to find r, but that would require the value of p which we had previously found. This is a mathematically legitimate strategy. However, using p to find r would result in a less accurate value of r, due to rounding error. We recommend using the given information as much as possible.



Example 4.3 Find y and z in the diagram.

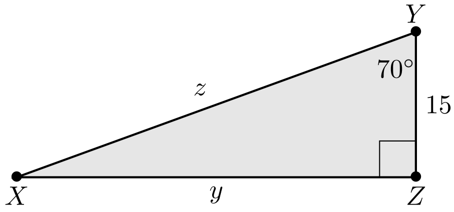
Solution Let us find y. We know the length of the leg opposite $\angle X$, and we want to find the length of the leg adjacent $\angle X$. The trigonometric function which relates these sides is tangent. We have

$$\tan 56^\circ = \frac{3}{y} \quad \text{implies} \quad y = \frac{3}{\tan 56^\circ} \approx 2.024.$$

All that is left is to find z. We know the length of the side opposite $\angle X$, and we want to find the hypotenuse. The trigonometric function which relates these sides is sine. In particular,

$$\sin 56^\circ = \frac{3}{z} \quad \text{implies} \quad z = \frac{3}{\sin 56^\circ} \approx 3.619.$$

■



Example 4.4 Find (a) the area and (b) the perimeter of $\triangle XYZ$.

Solution

(a) Recall that the area of a triangle is

$$\frac{1}{2}bh,$$

where b is the base and h is the height. If we treat x as the length of the height, then y is the length of the base. As result, our task is to find y. Notice

$$\tan 70^\circ = \frac{y}{15} \quad \text{implies} \quad y = 15\tan 70^\circ.$$

It follows that the area of $\triangle XYZ$ is

$$\frac{1}{2}\left(15\tan 70^\circ\right)(15) = \frac{225\tan 70^\circ}{2} \approx 309.091.$$

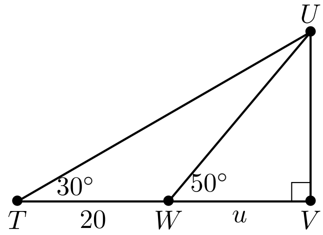
(b) We already know $y = 15\tan 70^\circ$, so all that we need is z. We have

$$\cos 70^\circ = \frac{15}{z} \quad \text{implies} \quad z = \frac{15}{\cos 70^\circ}.$$

Thus, the perimeter of $\triangle XYZ$ is

$$x + y + z = 15 + 15\tan 70^\circ + \frac{15}{\cos 70^\circ} \approx 100.069.$$

■



Example 4.5 What is the value of u?

Solution The idea is to use Theorem 4.1 on $\triangle UVW$ and $\triangle TUV$. The theorem allows us to write UV in terms of u two ways, which lets us establish an equation. Then we can solve for u.

In $\triangle UVW$, the length UV is opposite $\angle UWV$, and $VW = u$ is adjacent $\angle UWV$. The trigonometric function which relates these sides is tangent. Therefore,

$$\tan 50^\circ = \frac{UV}{u} \quad \text{implies} \quad UV = u \tan 50^\circ.$$

Consider $\triangle TUV$. We know UV is opposite $\angle T$, and $TV = u + 20$ is adjacent $\angle T$. Once again, tangent relates these sides. It follows that

$$\tan 30^\circ = \frac{UV}{u+2} \quad \text{implies} \quad UV = (u+20) \tan 30^\circ.$$

Since $UV = UV$, we have

$$\begin{aligned}
& u \tan 50^\circ &&= (u+20)\tan 30^\circ \\
\Rightarrow\quad & u \tan 50^\circ &&= u \tan 30^\circ + 20 \tan 30^\circ \\
\Rightarrow\quad & u \tan 50^\circ - u \tan 30^\circ &&= 20 \tan 30^\circ \\
\Rightarrow\quad & u(\tan 50^\circ - \tan 30^\circ) &&= 20 \tan 30^\circ \\
\Rightarrow\quad & u &&= \frac{20 \tan 30^\circ}{\tan 50^\circ - \tan 30^\circ} \\
& &&\approx 18.794.
\end{aligned}$$

■

Because there is a correspondence between radian measures and degree measures, evaluation of trigonometric functions at radian measures makes sense. The value of a trigonometric function at a radian measure is simply equal to the trigonometric function evaluated at the corresponding degree measure.

As a practical matter, evaluation of trigonometric functions at radian measures is simply a matter of adjusting your calculator to the correct mode.

Example 4.6 Evaluate.

(a) $\sin\frac{\pi}{5}$

(b) $12\cos\frac{3\pi}{7}$

(c) $5\tan\frac{\pi}{11}$

(d) $\sin 1$

Solution

(a) $\sin\frac{\pi}{5} \approx 0.588$

(b) $12\cos\frac{3\pi}{7} \approx 2.670$

(c) $5\tan\frac{\pi}{11} \approx 1.468$

(d) $\sin 1 \approx 0.841$

■

In Example 6 (d), there was no π in the expression. However, we know that we are evaluating a radian measure because there is no degree symbol.

In general, we advise readers to be mindful of the units. Using the incorrect setting changes the output. For example,

$$\tan 15^\circ \approx 0.268 \quad \text{and} \quad \tan 15 \approx -0.856.$$

The first result is tangent evaluated at the degree measure 15°, and the second is tangent evaluated at the radian measure 15. The latter corresponds to a degree measure of about 859.437°.

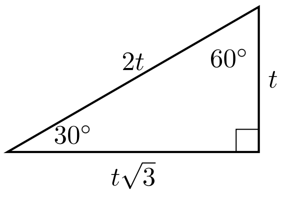
4.1.1 Trigonometric Functions and Special Right Triangles

We can use special right triangles to find the exact values of our three trigonometric functions evaluated at 30°, 45°, and 60°.

Proposition 4.1

θ	$30°$	$45°$	$60°$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Proof We will prove the first column, and leave the rest as exercises. Consider the 30°-60°-90° special right triangle.



Then

$$
\begin{aligned}
\sin 30° &= \frac{t}{2t}, & \cos 30° &= \frac{t\sqrt{3}}{2t}, \quad \text{and} & \tan 30° &= \frac{t}{t\sqrt{3}} \\
&= \frac{1}{2} & &= \frac{\sqrt{3}}{2} & &= \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\
& & & & &= \frac{\sqrt{3}}{3}.
\end{aligned}
$$

■

This leads us to make a corresponding radian version of Proposition 4.1.

Proposition 4.2

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Example 4.7 Evaluate.

(a) $\tan 45^\circ$

(b) $\sin\frac{\pi}{3}$

(c) $\cos 60^\circ$

(d) $\tan\frac{\pi}{6}$

Solution This is an application of Proposition 4.1 and Proposition 4.2.

(a) $\tan 45^\circ = 1$

(b) $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

(c) $\cos 60^\circ = \frac{1}{2}$

(d) $\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$

■

4.2 Inverse Trigonometric Functions

This section is an introduce to inverse trigonometric functions. In particular, we introduce arc sine, arc cosine, and arc tangent. To denote these functions evaluated at x, we write

$$\arcsin x, \quad \arccos x, \quad \text{and} \quad \arctan x,$$

respectively.

We will introduce a partial definition of arc sine, arc cosine, and arc tangent here. A more robust definition is contained in Chapter **??**.

Definition 4.1 Suppose that $0 \leq \theta \leq 90°$.

- The **arc sine** function $f(x) = \arcsin x$ is defined by the relationship
$$\arcsin x = \theta \quad \text{if} \quad \sin\theta = x$$
for $-1 \leq x \leq 1$.
- The **arc cosine** function $f(x) = \arccos x$ is defined by the relationship
$$\arccos x = \theta \quad \text{if} \quad \cos\theta = x$$
for $-1 \leq x \leq 1$.
- The **arc tangent** function $f(x) = \arctan x$ is defined by the relationship
$$\arctan x = \theta \quad \text{if} \quad \tan\theta = x$$
for $x \geq 0$.

There is alternative notation for these functions. In particular,

$$\sin^{-1} x = \arcsin x, \quad \cos^{-1} x = \arccos x, \quad \text{and} \quad \tan^{-1} x = \arctan x.$$

We prefer the "arc" notation, because it is less confusing. The -1 exponent in the alternative notation is sometimes incorrectly interpreted as the reciprocal of the trigonometric function. For example, sometimes students confuse $\sin^{-1} x$ and $1/\sin x$.

Example 4.8 Find the degree measure of each value.

(a) $\arcsin \dfrac{1}{2}$

(b) $\arccos 5$

(c) $\arctan 2$

(d) $\cos^{-1} \dfrac{2}{3}$

Solution

(a) $\arcsin \frac{1}{2} = 30°$

(b) $\arccos 5$ is undefined because 5 is not in the interval $[-1, 1]$.

(c) $\arctan 2 \approx 63.435°$

(d) $\cos^{-1} \frac{2}{3} \approx 48.190°$

■

Example 4.9 What is the radian measure of each value?

(a) $\sin^{-1} 7$

(b) $\arccos \frac{2}{5}$

(c) $\tan^{-1} \sqrt{3}$

(d) $\arcsin \frac{6}{11}$

Solution

(a) $\sin^{-1} 7$ is undefined because 7 is no in the interval $[-1, 1]$.

(b) $\arccos \frac{2}{5} \approx 1.159$ rad

(c) $\tan^{-1} \sqrt{3} \approx 1.047$ rad

(d) $\arcsin \frac{6}{11} \approx 0.577$ rad

■

Example 4.10 Suppose $\sin x = 0.7$, $\cos y = 8/11$, and $\tan z = 3$. Find the degree measure of (a) x, (b) y, and (c) z. Assume the angle measures are acute.

Solution We will utilize Definition 4.1 for (a), (b), and (c).

(a) Because $\sin x = 0.7$, we have $x = \arcsin 0.7 \approx 44.427°$.

(b) Due to the fact that $\cos y = 8/11$, the definition of arc cosine tells us $y = \arccos(8/11) \approx 43.342°$.

(c) Since $\tan z = 3$, we conclude $z = \arctan 3 \approx 71.565°$.

■

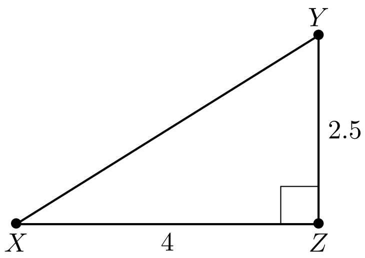
Problems like those in Example 10 could require radian measures as well. However, the only change in procedure is to convert your calculator to radian mode.

Example 4.11 Let $\cos A = 5/7$. Find the radian measure of $\angle A$.

Solution We make sure to change our calculator to radian mode. Then

$$\cos A = \frac{5}{7} \quad \text{implies} \quad m\angle A = \arccos\frac{5}{7} \approx 0.775 \text{ rad.}$$

■



Example 4.12 Find the degree measures of $\angle X$ and $\angle Y$.

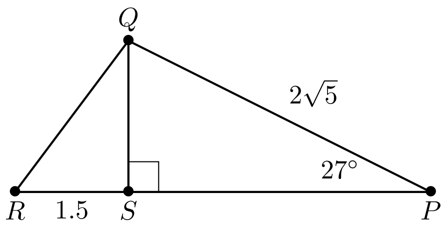
Solution Theorem 4.1 and Definition 4.1 tell us that

$$\tan X = \frac{2.5}{4} \quad \text{implies} \quad m\angle X = \arctan\frac{2.5}{4} \approx 32.005^\circ$$

and

$$\tan Y = \frac{4}{2.5} \quad \text{implies} \quad m\angle Y = \arctan\frac{4}{2.5} \approx 57.995^\circ.$$

■



Example 4.13 Find the degree measure of $\angle R$.

Solution Once we find QS, we can use arc tangent to find $m\angle R$. We know

$$QS = 2\sqrt{5}\sin 27^\circ \approx 2.030.$$

It follows that

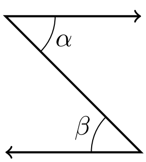
$$\tan R = \frac{2.030}{1.5} \quad \text{implies} \quad m\angle R \approx \arctan\frac{2.030}{1.5} \approx 53.543^\circ.$$

■

4.3 Angles of Elevation and Depression

Definition 4.2

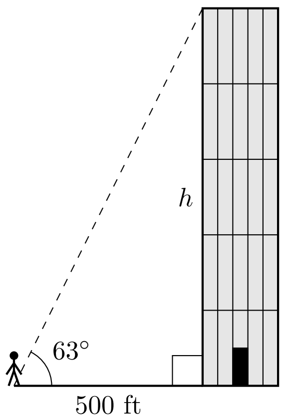
- An **angle of depression** is formed by a horizontal ray and another ray below it.
- An **angle of elevation** is formed by a horizontal ray and another ray above it.



In the diagram on page 13, α is an angle of depression and β is an angle of elevation.

Example 4.14 A man is standing 500 feet away from a tall building. If the angle of elevation to the top of the build from his perspective is $63°$, how tall is the building?

Solution The first step is to draw a diagram. We assume that the building is perpendicular to the horizontal; this assumption is standard practice when solving this type of problem. We neglect the height of the man, because the question did not provide it.

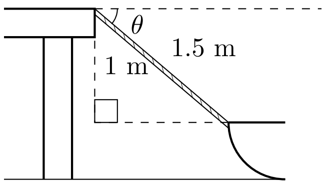


Let h be the height of the building. Then

$$\tan 63° = \frac{h}{500} \quad \text{implies} \quad h = 500 \tan 63° \approx 981.305.$$

Hence, the building is about 981 feet tall. ■

Example 4.15 A canoe is tethered to the floor of a dock by a rope of length 1.5 meters. The canoe is 1 meter below the floor of the dock. Calculate the angle of depression of the rope.

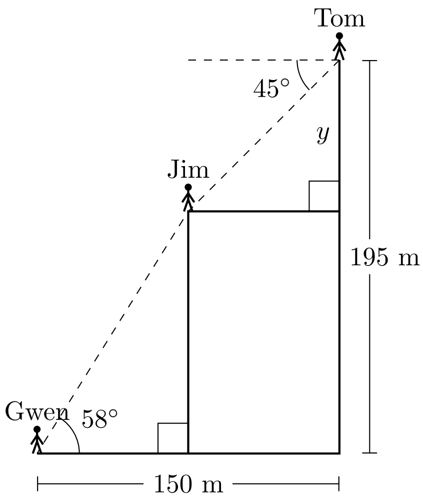


Solution Proposition **??** tells us that alternate interior angles are congruent. So, the angle opposite the 1-meter side in the triangle above has measure θ. Hence,

$$\sin\theta = \frac{1}{1.5} \quad \text{implies} \quad \theta = \arcsin\frac{1}{1.5} \approx 41.810^\circ.$$

Thus, the angle of depression is about 41.810°. ∎

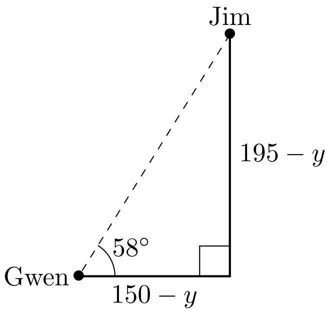
Example 4.16 Tom, Jim, and Gwen are hiking in a steep canyon 195 meters deep. Tom is just starting, Jim has been hiking for awhile but has not reached the bottom, and Gwen is already at the bottom. Tom looks down at an angle of depression of 45° to see Jim. Gwen looks up to see Jim at an angle of elevation of 58°. What is Jim's vertical distance from Tom, if Tom and Gwen are a horizontal distance of 150 meters apart and Jim is horizontally between Gwen and Tom?



Solution To find the vertical distance between Jim and Tom y, we will set up two right triangles which contain y. Then we can use trigonometry and algebra to find y.

The first right triangle we consider is the one which has Jim and Tom each standing at a vertex. Using the fact that alternate interior angles are congruent and the $45° - 45° - 90°$ special right triangle, we know the horizontal distance between Tom and Jim is also y.

The second triangle we consider is the one which has Gwen and Jim each standing at a vertex. It is clear from our previous work that the horizontal distance between Jim and Gwen is $150 - y$ and the vertical distance between Jim and Gwen is $195 - y$. This gives us the following triangle.



All that is left is a computation:

$$\begin{aligned}
& \tan 58^\circ &&= \frac{195 - y}{150 - y} \\
\Rightarrow\quad & (150 - y)\tan 58^\circ &&= 195 - y \\
\Rightarrow\quad & 150\tan 58^\circ - y\tan 58^\circ &&= 195 - y \\
\Rightarrow\quad & y - y\tan 58^\circ &&= 195 - 150\tan 58^\circ \\
\Rightarrow\quad & y(1 - \tan 58^\circ) &&= 195 - 150\tan 58^\circ \\
\Rightarrow\quad & y &&= \frac{195 - 150\tan 58^\circ}{1 - \tan 58^\circ} \\
& &&\approx 75.042.
\end{aligned}$$

We conclude that the vertical distance between Jim and Tom is about 75 meters. ■

4.4 Exercises

* Exercise 1

Evaluate. Round to three decimal places.

(a) $\tan 47°$

(b) $\sin 17°$

(c) $\sin 83°$

(d) $5\cos 42°$

(e) $6\tan 1°$

(f) $4\cos 15°$

* Exercise 2

Evaluate. Round to three decimal places.

(a) $\sin \frac{8\pi}{9}$

(b) $\tan \frac{\pi}{7}$

(c) $4\cos \frac{3\pi}{10}$

(d) $\tan 1.5$

(e) $2\sin \frac{7\pi}{9}$

(f) $5\cos 0.2$

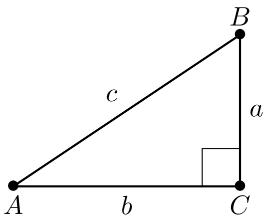


Figure 1

* Exercise 3

Consider Figure 1. Suppose $a = 3$, $b = 4$, and $c = 5$. Find the following.

(a) $\cos B$

(b) $\tan A$

(c) $\sin B$

(d) $\sin A$

(e) $\tan B$

(f) $\cos A$

* Exercise 4

Repeat Exercise 3, but assume $a = 2$, $b = 3$, and $c = \sqrt{13}$.

* Exercise 5

In Figure 1, say $m\angle A = 34°$. Use a calculator and the given side length to find the lengths of the remaining two sides.

(a) $a = 7$

(b) $b = 11$

(c) $c = 20$

(d) $a = \sqrt{7}$

* Exercise 6

Consider Figure 1. Suppose $m\angle B = 50°$. Use a calculator and the given side length to find the lengths of the remaining two sides.

(a) $a = 18$

(b) $b = 5$

(c) $c = 102$

(d) $b = \frac{7}{5}$

** Exercise 7

Use Figure 1 and the given information to find (i) the area and (ii) the perimeter of of $\triangle ABC$.

(a) Suppose $m\angle A = 25°$ and $c = 19$.

(b) Assume $m\angle B = 40°$ and $a = 25$.

(c) Say $m\angle A = 35°$ and $a = 10$.

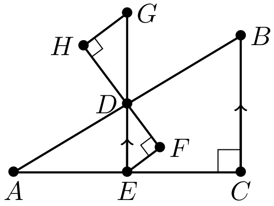


Figure 2

** Exercise 8

In Figure 2, assume $m\angle A = 40°$, $DE = 2$, $DF = DH$, and $\overline{BC}$ is parallel to $\overline{EG}$. Find each of the following.

(a) DF

(b) EF

(c) GH

(d) DG

** Exercise 9

Consider Figure 2. Let $m\angle B = 70°$, $GH = 5$, $DF = \frac{3}{4}DH$, and $\overline{BC}$ is parallel to $\overline{EG}$. What are each of the following?

(a) DG

(b) DH

(c) EF

(d) DE

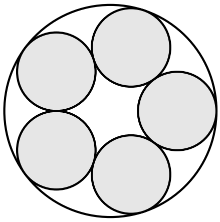


Figure 3

*** Exercise 10

In Figure 3, suppose the radius of the large circle is 12, and the five gray circles have the same radius. Find the total area of the gray circles. Hint: Tangent lines of circles are perpendicular to radii at points of tangency.

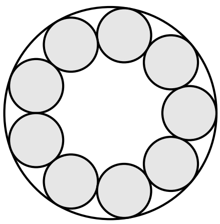


Figure 4

*** Exercise 11

Consider Figure 4. Say the radius of the large circle is 10, and the radii of the nine gray circles are equal. Calculate the ratio of the gray area to the white area. Hint: Tangent lines of circles are perpendicular to radii at points of tangency.

* Exercise 12

Find the exact value.

(a) $\sin 60°$

(b) $\cos 45°$

(c) $\tan 30°$

(d) $\cos 30°$

(e) $\sin 30°$

(f) $\tan 60°$

* Exercise 13

What is the exact value?

(a) $\sin \frac{\pi}{6}$

(b) $\tan \frac{\pi}{4}$

(c) $\cos \frac{\pi}{6}$

(d) $\tan \frac{\pi}{3}$

(e) $\cos \frac{\pi}{4}$

(f) $\sin \frac{\pi}{4}$

** Exercise 14

Prove the second and third column of Proposition 4.1.

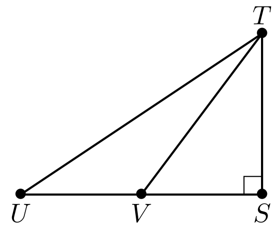


Figure 5

*** Exercise 15

In Figure 5, let $m\angle U = 40°$ and $m\angle SVT = 63°$.

(a) What is UV if $SV = 5$?

(b) Say $ST = 12$. Find UV.

(c) Assume $TU = 21$. What is TV?

(d) Find ST supposing $UV = 8$.

*** Exercise 16

Use Figure 5 and suppose $m\angle U = 35°$ and $m\angle STV = 75°$.

(a) Suppose $UV = 12$. Then TU equals what?

(b) Find UV if $SV = 12$.

(c) Say $UV = 2$. What is TV?

(d) Assume the area of $\triangle TUV$ is 300. What is SV?

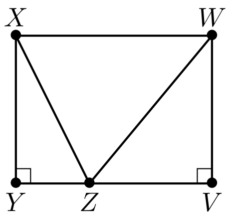


Figure 6

*** Exercise 17

Consider Figure 6. Suppose $m\angle VWZ = 40°$ and $m\angle YXZ = 27°$

(a) Assume $WX = 15$. Then XZ is equal to what?

(b) Find VY when $XY = 10$.

(c) Let the area of $\triangle VWZ$ be 33.984. What is the area of $\triangle XYZ$?

(d) Suppose the area of rectangle $VWXY$ is 113.962. Calculate the length WZ.

*** Exercise 18

Use Figure 6 to solve the following problems.

(a) Say $WX = 50$, $m\angle WXZ = 61°$, and $m\angle VZW = 25°$. Find XZ.

(b) Assume $XY = 18$, $m\angle VWZ = 42°$, and $m\angle YXZ = 29°$. What is the value of VY?

(c) Let $WX = 80$, $m\angle YXZ = 28°$, and $m\angle VWZ = 41°$. Then the length XY is what?

(d) Suppose the area of $\triangle XYZ$ is 4.807. If $m\angle YXZ = 31°$ and $m\angle XWZ = 44°$, calculate the area of $\triangle WXZ$.

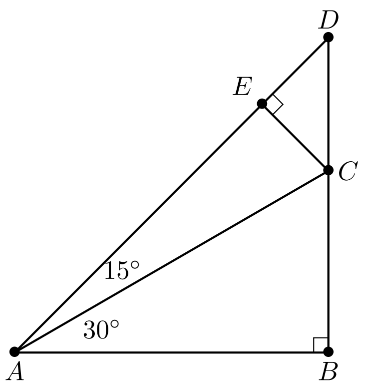


Figure 7

*** Exercise 19

With the aid of Figure 7, find the exact values of the following.

(a) $\sin 15°$

(b) $\cos 15°$

(c) $\tan 15°$

Hint: Notice that $m\angle BAD = 45°$ and $m\angle D = 45°$ and use special right triangles.

* Exercise 20

Find the degree measure. Some expressions are undefined.

(a) $\arcsin 0.2$

(b) $\cos^{-1}\sqrt{\frac{1}{7}}$

(c) $\arctan 5$

(d) $\sin^{-1} 4$

(e) $\arccos 0.25$

(f) $\tan^{-1}\frac{1}{3}$

* Exercise 21

What is the radian measure? Some expressions are undefined.

(a) $\sin^{-1}\frac{\sqrt{5}}{3}$

(b) $\arccos 15$

(c) $\tan^{-1}\frac{1}{2}$

(d) $\arcsin\frac{3}{\sqrt{3}}$

(e) $\cos^{-1} 0.7$

(f) $\arctan 15$

* Exercise 22

Find the *degree* measure of α. Assume that α is an acute angle.

(a) $\sin\alpha = \frac{3}{7}$

(b) $\cos\alpha = 0.2$

(c) $\tan\alpha = \frac{17}{13}$

(d) $\sin\alpha = 0.5$

(e) $\cos\alpha = \frac{\sqrt{2}}{2}$

(f) $\tan\alpha = \sqrt{3}$

* Exercise 23

Compute the *radian* measure of β. Assume that β is an acute angle.

(a) $\sin\beta = \frac{\sqrt{2}}{5}$

(b) $\cos\beta = \frac{5}{3\sqrt{7}}$

(c) $\tan\beta = 0.78$

(d) $\sin\beta = \frac{\sqrt{2}}{2}$

(e) $\cos\beta = \frac{\sqrt{3}}{2}$

(f) $\tan\beta = \frac{\sqrt{3}}{3}$

* Exercise 24

Use Figure 1 and the given information to find the degree measure of $\angle A$.

(a) $b = 10$ and $c = 11$

(b) $a = 3$ and $b = 5$

(c) $a = 1$ and $c = 7$

(d) $b = 5$ and $c = 9$

* Exercise 25

Consider Figure 1. Find the radian measure of $\angle B$, using the given information.

(a) $a = 2$ and $c = 5$

(b) $a = 7$ and $b = 15$

(c) $b = 5$ and $c = 10$

(d) $a = 2$ and $b = 3$

* Exercise 26

Use Propositions 4.1 and 4.2 to fill in the table with the appropriate (a) degree measures and (b) radian measures.

x	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\arcsin x$			
$\arccos x$			

* Exercise 27

Use Propositions 4.1 and 4.2 to find the exact solution to each equation. Write your answer (a) using degrees and (b) using radians.

- $\tan \alpha = 1$,
- $\tan \beta = \sqrt{3}$, and
- $\tan \gamma = \frac{\sqrt{3}}{3}$.

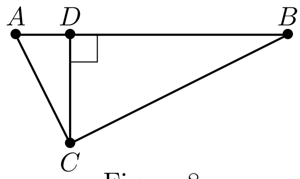


Figure 8

*** Exercise 28

Use the given information to find the degree measure of the angle in Figure 8.

(a) Suppose $m\angle B = 27°$, $BD = 4$, and $AC = \sqrt{5}$. Find $m\angle A$.

(b) Let $m\angle BCD = 55°$, $BC = 6\sqrt{10}$, and $AD = 3$. What is $m\angle ACD$?

(c) If $AB = 10$, $m\angle A = 61°$, what is $m\angle B$?

(d) Assume $m\angle A = 25°$, $BC = 4.5$, and $AC = 2\sqrt{5}$. Find $m\angle B$.

(e) Say $m\angle B = 45°$, $AB = 7$, and $AC = 5$. What is $m\angle A$, if its measure is greater than 45°?

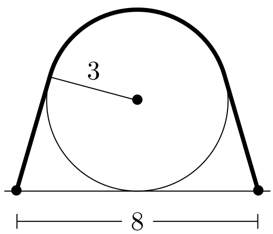


Figure 9

*** Exercise 29

Consider Figure 9. Calculate the length of the thick band. Hint: Tangent lines of circles are perpendicular to radii at points of tangency.

** Exercise 30

An observer sees a helicopter that is a horizontal distance of 1 mile from her location. The angle of elevation from the observer's perspective to the helicopter is 10°.

(a) What is the distance between the helicopter and the ground?

(b) Find the distance between the observer and the helicopter.

** Exercise 31

A yogi bends at the waist and has a straight back and neck. Suppose his back is bent to an angle of depression of 60°, and the yogi is 168 cm tall. The yogi's back and head comprise 50% of his total height.

(a) What is the distance between the tip of the yogi's head and his legs?

(b) How far is the tip of the yogi's head from the ground?

** Exercise 32

An airplane ascends from the ground at an angle of elevation of 25°. Suppose there are 200 feet of runway left when the plane lifts off.

(a) Calculate the distance the plane has traveled since lift off when it is at the end of the runway.

(b) How far is the plane from the ground when it is at the end of the runway?

** Exercise 33

A daredevil is constructing a ramp for a stunt. She designs the ramp to have a horizontal length of 5 feet and a vertical height of 3 feet.

(a) Find the slant length of the ramp.

(b) What is the angle of elevation of the ramp?

** Exercise 34

A young boy is inspecting a slide. The vertical ladder to the top of the slide is 4 feet. The boy estimates that the angle of depression from the top of the ladder to the bottom of the slide is 35°.

(a) Calculate the length of the slide.

(b) How far is the bottom of the ladder from the bottom of the slide?

(c) The boy observes that it takes about 0.729 seconds for a colleague to slide down. What was the colleague's average speed?

** Exercise 35

A straight 10-mile road goes up a hill. The change in elevation between the top and bottom of the hill is 0.5 miles. Calculate the angle of elevation of the road.

** Exercise 36

A tire swing hangs from a rope of length 10 feet. The rope is tied to a branch 12 feet from the ground. A girl pulls the tire back so that it is five feet from the ground. What is the angle of depression from the branch to the tire? Assume the rope is taut.

*** Exercise 37

An observer standing on a 500-meter building looks down at another building. The angle of depression from the observer to the top of the other building is 20° and the angle of depression to the bottom of the building is 50°.

(a) Calculate the height of the other building.

(b) Find the distance between the buildings.

*** Exercise 38

Heather and Sabrina are 300 feet apart and are on opposite sides of a tall tree. The angle of elevation of Heather to the top of the tree is 20° and the angle of elevation of Sabrina to the tree is 30°. How tall is the tree? Assume both women are 5.5 feet tall.

*** Exercise 39

The angle of elevation between Vaibhav and a mountain is 35°. He walks 200 meters closer to the mountain and his angle of elevation is 45°. Calculate the height of the mountain.

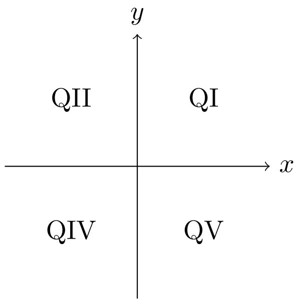
Chapter 5

Trigonometry of General Angles

This chapter defines the trigonometric functions we studied in Chapter 4. These definitions allow us to evaluate our trigonometric functions at angles of arbitrary measure, albeit only exactly at a limited number of values. We will introduce three more trigonometric functions as well: secant, cosecant, and cotangent. After our definitions are provided, the rest of the chapter will study properties of trigonometric functions. Inverse trigonometric functions will be discussed in Chapter **??**. Students will not need a calculator.

5.1 The Six Trigonometric Functions

The xy-coordinate system develops a correspondence between points and ordered pairs (x, y).



- Quadrant I (QI) is the set of points in the xy-plane such that $x > 0$ and $y > 0$.
- Quadrant II (QII) is the set of points in the xy-plane such that $x < 0$ and $y > 0$.
- Quadrant III (QIII) is the set of points in the xy-plane such that $x < 0$ and $y < 0$.
- Quadrant IV (QIV) is the set of points in the xy-plane such that $x > 0$ and $y < 0$.

Points on either of the two axes are not in any quadrant. For example, the point $(5, 0)$ does not belong to a quadrant; it is on the x-axis. Similarly, the point $(0, -2)$ is not in a quadrant; it is on the y-axis.

Example 5.1 State the quadrant or axis in which each of the points lies.

(a) $T(-4, 2)$

(b) $U(1, 3)$

(c) $V\left(\sqrt{8}, -1\right)$

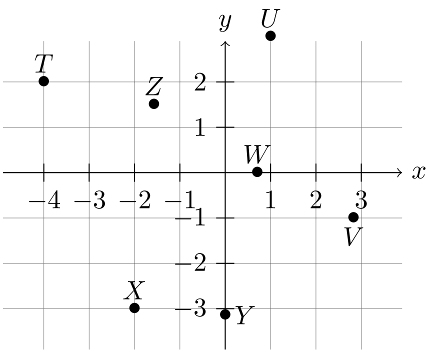
(d) $W\left(\dfrac{\sqrt{2}}{2}, 0\right)$

(e) $X(-2, -3)$

(f) $Y(0, -\pi)$

(g) $Z\left(-\dfrac{11}{7}, \dfrac{3}{2}\right)$

Solution Let us graph these points, so we can see their location. We will graph the points carefully. However, knowing whether each coordinate is positive, negative, or zero is enough to find the quadrant or axis in which the point lies.



From here, the conclusions follow easily:

(a) T is in quadrant II.

(b) U is in quadrant I.

(c) V is in quadrant IV.

(d) W is on the x-axis

(e) X is in quadrant III.

(f) Y is on the y-axis.

(g) Z is in quadrant II.

■

Definition 5.1 The **unit circle** is the set of points (x, y) of distance 1 from the origin. In other words, the unit circle is the set of points (x, y) such that

$$x^2 + y^2 = 1.$$

Our goal is to create a correspondence between directed angles and the unit circle. To achieve this, we will establish a convention for the location of directed angles' initial sides. This allows the position of the terminal side to be completely determined by the measure of the angle.

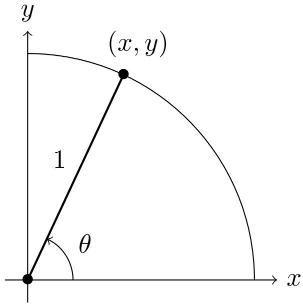
Definition 5.2 A **standard position angle** is an angle whose initial side lies on the positive x-axis.

Let us make the following convention:

> Assume all angles are in standard position when they are used within the context of trigonometric functions unless there is a reason to suppose otherwise.

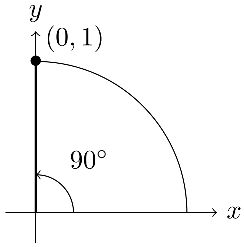
Because of this convention, we can be sloppy with our language. We often refer to angles simply by their measures, e.g. we say things like "angle 30°". We often say an angle "lies" in a particular quadrant; when we do this, it is understood that we are referring to the terminal side of a standard position angle.

We are ready to formulate our correspondence between standard position angles and points on the unit circle: Let the angle θ correspond to the point (x, y) where the terminal side of θ intersects the unit circle.

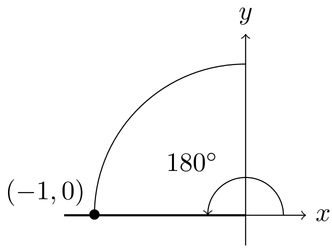


Example 5.2 Find the points on the unit circle corresponding to (a) $\theta = 90°$, (b) $\theta = 180°$, and (c) $\theta = 30°$.

Solution

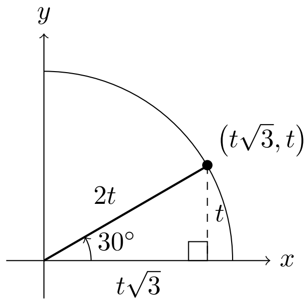


(a) As can be seen from the diagram above, the point corresponding to $\theta = 90°$ is $(0, 1)$.



(b) The diagram illustrates that $\theta = 180°$ corresponds to $(-1, 0)$.

(c) We need the $30° - 60° - 90°$ special right triangle.



Since the radius of the unit circle is 1, we have

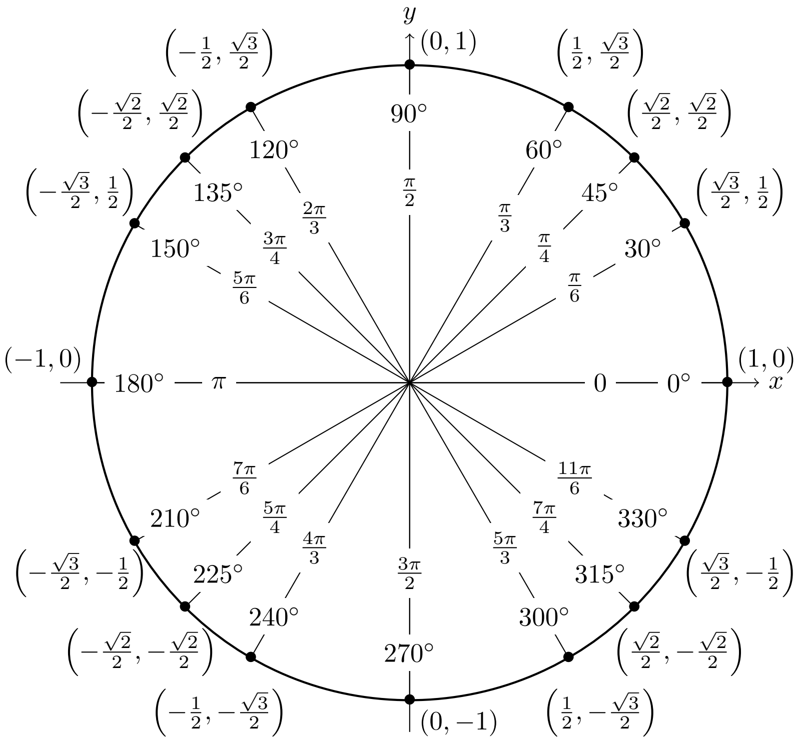
$$2t = 1 \quad \text{implies} \quad t = \frac{1}{2}.$$

Hence, the point corresponding to $\theta = 30°$ is $\left(\sqrt{3}/2, 1/2\right)$.

■

Using the 45°–45°–90° and 30°–60°–90° special right triangle we can fill in some key points in the first quadrant of the unit circle.

Then symmetry gives us points in the other three quadrants.



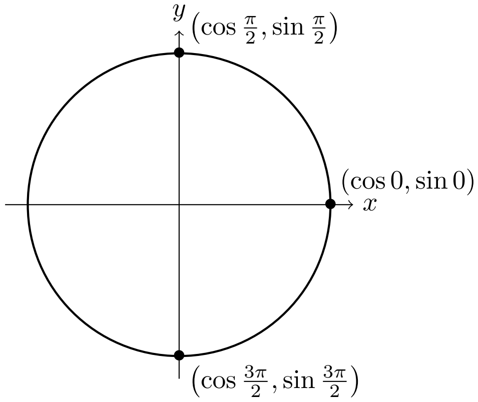
Definition 5.3 Suppose θ is an angle in standard position and (x, y) is the corresponding point on the unit circle. Define

$$\sin\theta = y, \quad \cos\theta = x, \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

A useful relationship that follows immediately from the above is that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Example 5.3 Find $\sin\theta$, $\cos\theta$, and $\tan\theta$ for (a) $\theta = 0$, (b) $\theta = \pi/2$, and (c) $\theta = 3\pi/2$ using the unit circle. When tangent is undefined say so.



Solution

(a) We know $\theta = 0$ corresponds to the point $(1, 0)$. Therefore,

$$\sin 0 = 0, \quad \cos 0 = 1 \quad \text{and} \quad \tan 0 = \frac{0}{1} = 0.$$

(b) Since $\theta = \pi/2$ corresponds to the point $(0, 1)$,

$$\sin\frac{\pi}{2} = 1 \quad \text{and} \quad \cos\frac{\pi}{2} = 0.$$

Tangent is undefined at $\pi/2$, because $\tan\theta = y/x$ and $x = 0$.

(c) Because $\theta = 3\pi/2$ corresponds to the point $(0, -1)$, we have

$$\sin\frac{3\pi}{2} = -1 \quad \text{and} \quad \cos\frac{3\pi}{2} = 0.$$

Tangent is undefined at $3\pi/2$, because $\tan\theta = y/x$ and $x = 0$.

■

We can utilize the unit circle on page 32 to evaluate trigonometric functions at more sophisticated angle measures.

Example 5.4 Find the exact value of each of the following.
(a) $\sin 120°$, (b) $\cos \dfrac{5\pi}{4}$, and (c) $\tan 330°$.

Solution

(a) We see that the point corresponding to $120°$ is $\left(-1/2, \sqrt{3}/2\right)$. Sine is the y-coordinate of this point. Hence,

$$\sin 120° = \frac{\sqrt{3}}{2}.$$

(b) The point corresponding to $5\pi/4$ is $\left(-\sqrt{2}/2, -\sqrt{2}/2\right)$. Cosine is the x-coordinate of this point. Thus,

$$\cos \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}.$$

(c) The point corresponding to $330°$ is $\left(\sqrt{3}/2, -1/2\right)$. Tangent is the ratio of the y- and x-coordinates. It follows that

$$\tan 330° = \frac{-1/2}{\sqrt{3}/2} = -\frac{\sqrt{3}}{3}.$$

■

Example 5.5 Suppose

$$\cos\theta = -\frac{\sqrt{3}}{2}.$$

Find all θ that satisfy this equation for $0 \le \theta < 2\pi$.

Solution If $\cos\theta = -\sqrt{3}/2$, then the points corresponding to θ on the unit circle must satisfy $x = -\sqrt{3}/2$. Via inspection of the unit circle, we see that if the x-coordinate of the point is $-\sqrt{3}/2$ and $0 \le \theta < 2\pi$, then

$$\theta = \frac{5\pi}{6} \quad \text{or} \quad \theta = \frac{7\pi}{6}.$$

$\blacksquare$

Definition 5.4 Suppose θ is an angle in standard position whose terminal side intersects the unit circle at (x, y). Define

$$\sec\theta = \frac{1}{x}, \quad \csc\theta = \frac{1}{y}, \quad \text{and} \quad \cot\theta = \frac{x}{y}.$$

Definition 5.5 An **identity** is a statement of equality between mathematical expressions, which holds for all values of the variables contained within the domains of each expression.

Identities are the subject of Chapter 7. However, we will introduce our first few in this chapter.

Proposition 5.1 (Reciprocal Identities) *For θ a degree or radian measure each expression holds whenever it is defined.*

- $\sec\theta = \dfrac{1}{\cos\theta}$
- $\csc\theta = \dfrac{1}{\sin\theta}$
- $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$
- $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

This proposition follows from simple substitutions, so we omit a formal proof.

Example 5.6 Evaluate each of the follow.
(a) $\sec\dfrac{7\pi}{4}$, (b) $\csc 60°$, and (c) $\cot\dfrac{5\pi}{6}$.

Solution

(a)

$$\begin{aligned}\sec\frac{7\pi}{4} &= \frac{1}{\cos(7\pi/4)} \\ &= \frac{1}{\sqrt{2}/2} \\ &= \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ &= \sqrt{2}.\end{aligned}$$

(b)

$$\begin{aligned}\csc 150^\circ &= \frac{1}{\sin 150^\circ}\\ &= \frac{1}{1/2}\\ &= 2.\end{aligned}$$

(c)

$$\begin{aligned}\cot\frac{5\pi}{6} &= \frac{\cos(5\pi/6)}{\sin(5\pi/6)}\\ &= \frac{-\sqrt{3}/2}{1/2}\\ &= -\sqrt{3}.\end{aligned}$$

■

Example 5.7 Assume $0 \le \varphi < 360^\circ$.

$$\cot\varphi = -\sqrt{3}.$$

Solve for φ.

Solution We know that cotangent of φ is the ratio of the x- and y-coordinates of the points corresponding to φ on the unit circle. To help us identify the appropriate points, note that

$$-\sqrt{3} = -\frac{\sqrt{3}/2}{1/2}.$$

Then, via inspection of the unit circle, we see the ratio of the x- and y-coordinates of either $\left(-\sqrt{3}/2, 1/2\right)$ or $\left(\sqrt{3}/2, -1/2\right)$ produces a quotient of $-\sqrt{3}$. Angles 150° and 330° correspond to the points $\left(-\sqrt{3}/2, 1/2\right)$ and $\left(\sqrt{3}/2, -1/2\right)$, respectively. Hence,

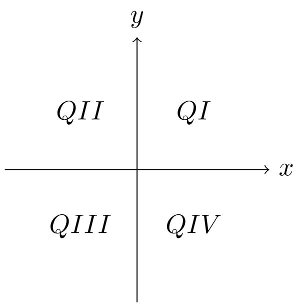
$$\varphi = 150^\circ \quad \text{or} \quad \varphi = 330^\circ.$$

■

5.2 Reference Angles

Committing the entire unit circle to memory is a challenge for many students. In this section, we will develop techniques to dramatically reduce the necessary amount of memorization.

Let us begin with a discussion of the sign of sine, cosine, and tangent. Cosine and sine correspond to the x- and y-coordinates on the unit circle, respectively, so their signs depend on the quadrant in which the terminal side of the angle lies.



The following table lists the quadrants in which sine, cosine, and tangent are positive and negative.

Quadrant	Positive	Negative
I	sine, cosine, and tangent	none
II	sine	cosine and tangent
III	tangent	sine and cosine
IV	cosine	sine and tangent

Many students remember this using a mnemonic device. For example,

> **A** **S**mart **T**rig **C**lass
>
> **A**ll trigonometric functions are positive in quadrant I.
> **S**ine is positive in quadrant II.
> **T**angent is positive in quadrant III.
> **C**osine is positive in quadrant IV.

Example 5.8 Determine in which quadrant θ lies via the signs of the given trigonometric functions.

(a) $\cos\theta > 0$ and $\sin\theta < 0$

(b) $\tan\theta > 0$ and $\csc\theta < 0$

(c) $\cot\theta < 0$ and $\sec\theta > 0$

Solution

(a) Since $\cos\theta > 0$, we know θ lies in quadrant I or IV. Because $\sin\theta < 0$, we conclude θ lies in quadrant III or IV. The only quadrant held in common is quadrant IV. Hence, θ lies in quadrant IV.

(b) Since $\tan\theta > 0$, it follows that θ lies in quadrant I or III. Due to the fact that $\csc\theta < 0$ implies $\sin\theta < 0$, it must be the case that θ lies in quadrant III or IV. By the process of elimination, θ lies in quadrant III.

(c) If $\cot\theta < 0$, then $\tan\theta < 0$, which means that θ lies in quadrant II or IV. Because $\sec\theta > 0$ is the same as saying $\cos\theta > 0$, we know θ lies in quadrant I or IV. Thus, θ must lie in quadrant IV. ■

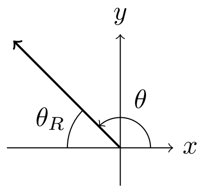
Definition 5.6 Consider the directed angle θ in standard position. Suppose the terminal side of θ lies within a quadrant. The **reference angle** θ_R is the acute angle formed by the terminal side of θ and either the positive or negative x-axis.

Note that the acuteness of θ_R determines whether the positive or negative x-axis forms a side of θ_R. Only the closer of the two makes an acute angle with the terminal side of θ.

We can find simple formulas for θ_R, if we suppose $0 < \theta < 2\pi$. When the terminal side of θ is in quadrant I, the closest x-axis is the positive one. So, the measure of the reference angle is

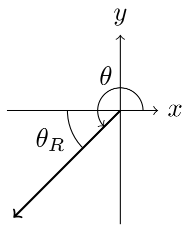
$$\theta_R = \theta.$$

The other three quadrants are a bit more complicated.



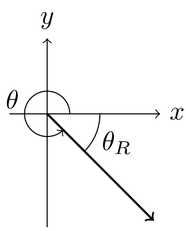
When the terminal side of θ is in quadrant II, θ_R is the angle formed by the negative x-axis and terminal side of θ. Its measure is

$$\theta_R = \pi - \theta.$$



When the terminal side of θ is in quadrant III, θ_R is the angle formed by the negative x-axis and the terminal side of θ. Its measure is

$$\theta_R = \theta - \pi.$$



When the terminal side of θ is in quadrant IV, θ_R is the angle formed by the positive x-axis and the terminal side of θ. Its measure is

$$\theta_R = 2\pi - \theta.$$

Proposition 5.2 summarizes the above.

Proposition 5.2 *For an angle θ of radian measure between 0 and 2π or of degree measure between 0 and $360°$, the following table provides the measure of the reference angle.*

θ_R *of* θ	*QI*	*QII*	*QIII*	*QIV*
Radian measure	θ	$\pi - \theta$	$\theta - \pi$	$2\pi - \theta$
Degree measure	θ	$180° - \theta$	$\theta - 180°$	$360° - \theta$

The six trigonometric functions evaluated at θ and θ_R have the same magnitude. So, to evaluate a trigonometric function at an angle measure, evaluate it at its reference angle and change the sign as needed.

Example 5.9 Evaluate

(a) $\tan \frac{7\pi}{4}$

(b) $\sin 120°$

(c) $\csc 210°$

(d) $\cos \frac{5\pi}{3}$

Solution

(a) Since

$$\frac{3\pi}{2} < \frac{7\pi}{4} < 2\pi,$$

$7\pi/4$ lies in quadrant IV. This implies that tangent is negative. Furthermore, the reference angle is

$$\theta_R = 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}.$$

Since $\tan(\pi/4) = 1$, we have

$$\tan \frac{7\pi}{4} = -\tan \frac{\pi}{4} = -1.$$

(b) Due to the fact that

$$90^\circ < 120^\circ < 180^\circ,$$

120° lies in quadrant II. Sine is positive in quadrant II. Furthermore, the reference angle is

$$\theta_R = 180^\circ - 120^\circ = 60^\circ.$$

We know $\sin 60^\circ = \sqrt{3}/2$. Therefore,

$$\sin 120^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}.$$

(c) The angle 210° lies in quadrant III, because

$$180^\circ < 210^\circ < 270^\circ.$$

It follows that sine is negative, which means its reciprocal cosecant is also negative. Furthermore, the reference angle is

$$\theta_R = 210^\circ - 180^\circ = 30^\circ.$$

We have $\sin 30^\circ = 1/2$. So,

$$\begin{aligned}\csc 210^\circ &= -\csc 30^\circ \\ &= -\frac{1}{\sin 30^\circ} \\ &= -\frac{1}{1/2} \\ &= -2.\end{aligned}$$

(d) The inequality

$$\frac{3\pi}{2} < \frac{5\pi}{3} < 2\pi,$$

tells us that $5\pi/3$ lies in quadrant IV. As a result, we expect the output to be positive, because cosine is positive in quadrant IV. Furthermore, the reference angle is

$$\theta_R = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3}.$$

Because $\cos(\pi/3) = 1/2$,

$$\cos\frac{5\pi}{3} = \cos\frac{\pi}{3} = \frac{1}{2}.$$

■

5.3 More Evaluation Techniques

In this section, we will learn to evaluate trigonometric functions for $\theta \geq 360°$ and $\theta < 0$.

Definition 5.7 The function f is **periodic** with period $p > 0$ if p is the smallest number such that

$$f(x + p) = f(x)$$

for all x in the domain of f.

Proposition 5.3 *All six trigonometric functions are periodic. Their periods are given in the table below.*

	Degree period	*Radian period*
$\cos x$	$360°$	2π
$\sin x$	$360°$	2π
$\tan x$	$180°$	π
$\sec x$	$360°$	2π
$\csc x$	$360°$	2π
$\cot x$	$180°$	π

Using Proposition 5.3, it is not difficult to see that the output of a trigonometric function is not affected by adding or subtracting integer multiples of its period to the input. We can use this fact to convert the input into a value between 0 and $360°$, and then utilize the techniques discussed previously to evaluate.

Example 5.10 Evaluate.

(a) $\cot(-1710°)$

(b) $\cos 5\pi$

(c) $\tan\left(-\frac{4\pi}{3}\right)$

(d) $\csc 585°$

Solution

(a) Cotangent has period 180° when evaluated using degrees, and $-1710 \div 180 = -9\frac{1}{2}$. So, we will add $180°(10) = 1800°$ to the input:

$$\cot(-1710°) = \cot(-1710° + 1800°) = \cot 90° = 0.$$

(b) Cosine has period 2π when evaluated using radians, and $5\pi \div (2\pi) = 2\frac{1}{2}$. As a result, we will subtract $2\pi(2) = 4\pi$ from the input:

$$\cos 5\pi = \cos(5\pi - 4\pi) = \cos\pi = -1.$$

(c) Tangent has period π when evaluated using radians, and $-\frac{4\pi}{3} \div \pi = -1\frac{1}{3}$. To make the number inside positive, we will add $\pi(2) = 2\pi$ to the input:

$$\tan\left(-\frac{4\pi}{3}\right) = \tan\left(-\frac{4\pi}{3} + 2\pi\right) = \tan\frac{2\pi}{3}.$$

Then we will use reference angles to evaluate:

$$\tan\frac{2\pi}{3} = -\tan\frac{\pi}{3} = -\frac{\sqrt{3}}{3}.$$

Hence,

$$\tan\left(-\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{3}.$$

(c) Cosecant has period 360° when evaluated using degrees, and $585° \div 360° = 1\frac{5}{8}$. Subtracting $360°(1) = 360°$ from the input makes the computation more tractable:

$$\csc 585° = \csc(585° - 360°) = \csc 225°.$$

Using reference angles, we have

$$\begin{aligned}\csc 225^\circ &= -\csc 45^\circ \\ &= -\frac{1}{\sin 45^\circ} \\ &= -\frac{1}{\sqrt{2}/2} \\ &= -\sqrt{2}.\end{aligned}$$

Therefore,

$$\csc 585^\circ = -\sqrt{2}.$$

■

Because trigonometric functions are periodic, if θ is a solution then so is θ plus or minus an integer multiple of the period. With this in mind, we are ready to handle general solutions of trigonometric equations.

Example 5.11 Suppose

$$\sec\theta = 2.$$

Find all values of θ.

Solution We know

$$\sec\theta = 2 \quad \text{implies} \quad \cos\theta = \frac{1}{2}.$$

Because $\cos(\pi/3) = 1/2$, the reference angle is $\pi/3$. Cosine is positive when θ lies in quadrant I or IV. Hence, the solutions for $0 \le \theta < 2\pi$ are

$$\theta = \frac{\pi}{3} \quad \text{and} \quad \theta = \frac{5\pi}{3}.$$

Since secant is periodic with period 2π, any integer multiple of 2π added to either result will produce $1/2$. Thus, the solutions are

$$\theta = \frac{\pi}{3} + 2\pi n \quad \text{and} \quad \theta = \frac{5\pi}{3} + 2\pi n$$

for $n = 0, 1, -1, 2, -2, \ldots$. ■

Example 5.12 Solve for θ.

$$\sin(3\theta) = -\frac{\sqrt{2}}{2}.$$

Suppose $0 \leq \theta < 2\pi$.

Solution If $0 \leq \theta < 2\pi$, then $0 \leq 3\theta < 6\pi$. Using what we know about the unit circle, the solutions for 3θ in the interval $[0, 2\pi)$ are the values such that $3\theta = 5\pi/4$ and $3\theta = 7\pi/4$. To find the solutions in the interval $[2\pi, 6\pi)$ note that adding $2\pi = 8\pi/4$ to a previous solution produces another solution, as long as the sum remains in the interval. Using this insight, we have

$$3\theta = \frac{5\pi}{4}, \ \frac{7\pi}{4}, \ \frac{13\pi}{4}, \ \frac{15\pi}{4}, \ \frac{21\pi}{4}, \ \text{and} \ \frac{23\pi}{4}$$

are the values of 3θ in the interval $[0, 6\pi)$ which satisfy the equation. Dividing by 3 yields our solutions:

$$\theta = \frac{5\pi}{12}, \ \frac{7\pi}{12}, \ \frac{13\pi}{12}, \ \frac{5\pi}{4}, \ \frac{7\pi}{4}, \ \text{and} \ \frac{23\pi}{12}.$$

■

Definition 5.8

- The function f is **even** if

$$f(-x) = f(x).$$

- The function f is **odd** if

$$f(-x) = -f(x).$$

Proposition 5.4 (Even and Odd Identities)

(i) The function $\sin\theta$ *is odd, so*

$$\sin(-\theta) = -\sin\theta.$$

(ii) The function $\cos\theta$ *is even, so*

$$\cos(-\theta) = \cos\theta.$$

(iii) The function $\tan\theta$ *is odd, so*

$$\tan(-\theta) = -\tan\theta.$$

(iv) The function $\sec\theta$ *is even, so*

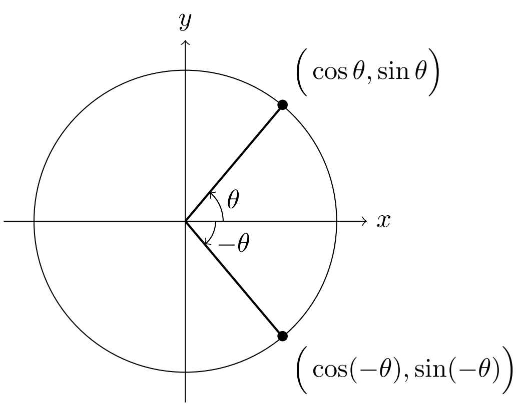
$$\sec(-\theta) = \sec\theta.$$

(v) The function $\csc\theta$ *is odd, so*

$$\csc(-\theta) = -\csc\theta.$$

(vi) The function $\cot\theta$ *is odd, so*

$$\cot(-\theta) = -\cot\theta.$$



Proof Examination of the unit circle and the definitions of sine and cosine leads us to conclude that sine and cosine are odd and even, respectively. So, (i) and (ii) hold.

We will prove (iii) and (iv), and leave the rest as exercises.

(iii)

$$\begin{aligned}\tan(-x) &= \frac{\sin(-x)}{\cos(-x)}\\ &= \frac{-\sin x}{\cos x}\\ &= -\tan x.\end{aligned}$$

(iv)

$$\begin{aligned}\sec(-x) &= \frac{1}{\cos(-x)}\\ &= \frac{1}{\cos x}\\ &= \sec x.\end{aligned}$$

■

Example 5.13 Evaluate (a) $\sin(-30^\circ)$ and (b) $\sec\left(-\frac{5\pi}{6}\right)$.

Solution

(a) Because sine is odd,

$$\sin(-30^\circ) = -\sin 30^\circ = -\frac{1}{2}.$$

(b) Since secant is even,

$$\begin{aligned}\sec\left(-\frac{5\pi}{6}\right) &= \sec\frac{5\pi}{6}\\ &= -\sec\frac{\pi}{6}\\ &= -\frac{1}{\cos(\pi/6)}\\ &= -\frac{1}{\sqrt{3}/2}\\ &= -\frac{2\sqrt{3}}{3}.\end{aligned}$$

■

5.4 Finding the Values of Trigonometric Functions

In this section, we will study how to use the value of one trigonometric function to find the values of the other five, e.g. we are given $\sin\theta$ and we will study how to find $\cos\theta$, $\tan\theta$, $\sec\theta$, etc. Surprisingly, a helpful approach to solving this type of problem is to consider where the terminal side of θ intersects a particular circle.

Our next theorem will be of great utility throughout the rest of this book. To understand the theorem, recall that a circle centered at $(0,0)$ and of radius $r > 0$ has equation

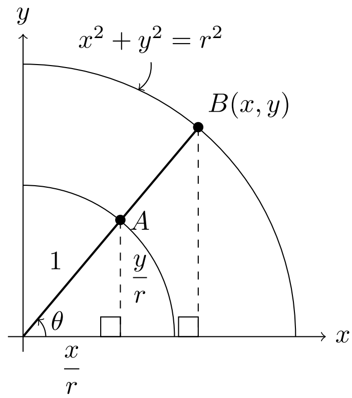
$$x^2 + y^2 = r^2.$$

Theorem 5.1 *Consider the circle with equation*

$$x^2 + y^2 = r^2$$

where $r > 0$. *Let* (x,y) *be a point on the circle, and suppose* θ *is the standard position angle which includes* (x,y) *on its terminal side. Then*

- $\sin\theta = \dfrac{y}{r}$
- $\cos\theta = \dfrac{x}{r}$
- $\tan\theta = \dfrac{y}{x}$
- $\sec\theta = \dfrac{r}{x}$
- $\csc\theta = \dfrac{r}{y}$
- $\cot\theta = \dfrac{x}{y}$



Proof A quick check shows that the identities hold for (x, y) on the x- or y-axis. Suppose the point (x, y) is contained in a quadrant. Then construct right triangles like in the diagram above. Using similar triangles, we have that the sides of the right triangle with a hypotenuse of length 1 are $1/r$ times the lengths of the sides of the right triangle with a hypotenuse of length r. Furthermore, the signs in each coordinate of A and B agree because both points lie in the same quadrant. We conclude that the terminal side of θ intersects the unit circle at $(x/r, y/r)$. Hence, the definitions of sine, cosine, and tangent tell us that

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \text{and} \quad \tan\theta = \frac{y}{x}.$$

The other ratios follow from Proposition 5.1.

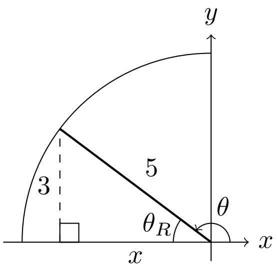
■

Example 5.14 Suppose

$$\sin\theta = \frac{3}{5} \quad \text{and} \quad \sec\theta < 0.$$

Find the values of the remaining five trigonometric functions.

Solution Because $\sin\theta > 0$ and $\sec\theta < 0$, the angle θ is in quadrant II. Let us suppose that we have a circle of radius 5.



The Pythagorean Theorem (Theorem **??**) tells us

$$x^2 + 3^2 = 5^2,$$

which implies $x = 4$ or $x = -4$. Since our triangle is in quadrant II, $x = -4$. From here, Theorem 5.1 tells us

$$\cos\theta = -\frac{4}{5}, \quad \tan\theta = -\frac{3}{4}, \quad \sec\theta = -\frac{5}{4},$$
$$\csc\theta = \frac{5}{3}, \quad \text{and} \quad \cot\theta = -\frac{4}{3}.$$

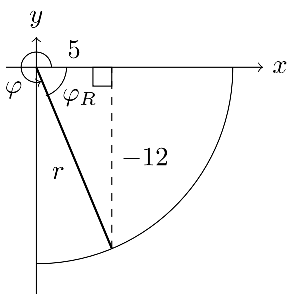
■

Example 5.15 Suppose

$$\tan\varphi = -\frac{12}{5} \quad \text{and} \quad \cos\varphi > 0.$$

Find the values of the remaining five trigonometric functions.

Solution Since $\tan\varphi < 0$ and $\cos\varphi > 0$, the terminal side of φ lies in quadrant IV. Suppose the signed length of the side opposite φ is -12. Then the side adjacent has length 5.



Using the Pythagorean Theorem (Theorem **??**) we have

$$r^2 = 5^2 + (-12)^2 \quad \text{implies} \quad r = \pm 13.$$

The hypotenuse is always positive, so $r = 13$. Thus, the values of the five remaining trigonometric functions are the following.

$$\begin{aligned} \sin\theta &= -\frac{12}{13}, & \cos\theta &= \frac{5}{13}, & \sec\theta &= \frac{13}{5}, \\ \csc\theta &= -\frac{13}{12}, & &\text{and} & \cot\theta &= -\frac{5}{12}. \end{aligned}$$

■

5.5 Pythagorean Identities

Let us introduce some important notation. When we write

$$\sin^2\theta, \quad \cos^2\theta, \quad \tan^2\theta, \quad \text{etc.},$$

we mean

$$(\sin\theta)^2, \quad (\cos\theta)^2, \quad (\tan\theta)^2, \quad \text{etc.},$$

respectively. That is, the notation $\sin^2\theta$, $\cos^2\theta$, $\tan^2\theta$, etc. means evaluate the trigonometric function at θ and then square the result.

In contract, the notation $\sin\theta^2$, $\cos\theta^2$, $\tan\theta^2$, etc. means square θ and then evaluate the result.

Example 5.16 Evaluate $\tan^2\left(\frac{\pi}{3}\right)$ and $\tan\left(\frac{\pi}{3}\right)^2$. Use a calculator where necessary.

Solution

$$\begin{aligned}\tan^2\left(\frac{\pi}{3}\right) &= \left(\tan\frac{\pi}{3}\right)^2\\ &= \left(\sqrt{3}\right)^2\\ &= 3\end{aligned}$$

and

$$\tan\left(\frac{\pi}{3}\right)^2 = \tan\frac{\pi^2}{9} \approx 1.948.$$

■

Theorem 5.2 (Pythagorean Identities)

For any real number θ the following equations hold whenever they are defined.

(i) $\cos^2\theta + \sin^2\theta = 1$

(ii) $1 + \tan^2\theta = \sec^2\theta$

(iii) $1 + \cot^2\theta = \csc^2\theta$

Proof

(i) Since the equation $x^2 + y^2 = 1$ is the unit circle, $x = \cos\theta$, and $y = \sin\theta$, the identity

$$\cos^2\theta + \sin^2\theta = 1$$

follows due to substitution.

(ii) Let us prove

$$1 + \tan^2\theta = \sec^2\theta.$$

Consider (i) and divide both sides by $\cos^2\theta$:

$$\begin{aligned} 1 + \frac{\sin^2\theta}{\cos^2\theta} &= \frac{1}{\cos^2\theta} \\ \Rightarrow \quad 1 + \left(\frac{\sin\theta}{\cos\theta}\right)^2 &= \left(\frac{1}{\cos\theta}\right)^2 \\ \Rightarrow \quad 1 + \tan^2\theta &= \sec^2\theta. \end{aligned}$$

(iii) Let us prove

$$1 + \cot^2\theta = \csc^2\theta.$$

Once again, consider (i). Divide both sides by $\sin^2\theta$:

$$\begin{aligned} \frac{\cos^2\theta}{\sin^2\theta} + 1 &= \frac{1}{\sin^2\theta} \\ \Rightarrow \quad 1 + \left(\frac{\cos\theta}{\sin\theta}\right)^2 &= \left(\frac{1}{\sin\theta}\right)^2 \\ \Rightarrow \quad 1 + \cot^2\theta &= \csc^2\theta. \end{aligned}$$

■

Example 5.17 Solve

$$5\sin x - 2\cos^2 x = 1.$$

Find all values of x.

Solution Our goal is to rewrite the equation so that its only trigonometric function is $\sin\theta$. To achieve this goal, we will use Pythagorean Identity (i) to convert $\cos^2\theta$ into an expression of $\sin\theta$. In particular, we have

$$\cos^2\theta = 1 - \sin^2\theta.$$

Then we will use substitution to rewrite our equation:

$$5\sin x - 2\cos^2 x = 5\sin x - 2(1 - \sin^2 x) = 2\sin^2 x + 5\sin x - 2.$$

It follows that

$$2\sin^2 x + 5\sin x - 3 = 0 \quad \text{implies} \quad (2\sin x - 1)(\sin x + 3) = 0$$

So,

$$\sin x = \frac{1}{2} \quad \text{or} \quad \sin x = -3.$$

The latter is an impossibility because sine is the y-coordinate on the unit circle and points on the unit circle have a y-coordinates between -1 and 1, inclusive. The equation

$$\sin x = \frac{1}{2}$$

has solutions $x = \pi/6$ and $x = 5\pi/6$ for $0 \leq x < 2\pi$. Since sine is periodic with period 2π any integer multiple of 2π added to either result is also a solution. Thus,

$$x = \frac{\pi}{6} + 2\pi n \quad \text{or} \quad x = \frac{5\pi}{6} + 2\pi n$$

for $n = 0, 1, -1, 2, -2, \ldots$. ■

5.6 Verifying Identities

In this section, we will verify trigonometric identities. This requires knowledge of the identities we have already discussed. We suggest readers review the Reciprocal Identities (Proposition 5.1), the periods of the trigonometric functions (Proposition 5.3), the Even and Odd Identities (Proposition 5.4), and the Pythagorean Identities (Theorem 5.2). They will be used frequently within this section and the corresponding exercises.

When you are asked to verify an identity consider one side of the equation and perform operations on it until the expression is identical to the other side of the equation.

Example 5.18 Verify the identity.

$$\cos(-\theta)\tan(-\theta) = -\sin\theta.$$

Solution We know that cosine is even and tangent is odd, so $\cos(-\theta) = \cos\theta$ and $\tan(-\theta) = -\tan\theta$. Furthermore,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Let us start with the left side and work our way to the right.

$$\begin{aligned}
\cos(-\theta)\tan(-\theta) &= \cos\theta\Big(-\tan\theta\Big) \\
&= -\cos\theta \cdot \frac{\sin\theta}{\cos\theta} \\
&= -\frac{\cos\theta}{1} \cdot \frac{\sin\theta}{\cos\theta} \\
&= -\frac{\sin\theta}{1} \\
&= -\sin\theta.
\end{aligned}$$

■

Example 5.19 Verify the identity.

$$\frac{\cos^2 x}{1 - \sin x} = 1 + \sin x.$$

Solution We will start on the left side, and use the Pythagorean Identity

$$\cos^2 x = 1 - \sin^2 x.$$

To do this we will multiply the top and bottom of the ratio by $1 + \sin x$ and use the difference of two squares formula:

$$\begin{aligned}
\frac{\cos^2 x}{1 - \sin x} &= \frac{\cos^2 x}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x} \\
&= \frac{\cos^2 x(1 + \sin x)}{1 - \sin^2 x} \\
&= \frac{\cos^2 x(1 + \sin x)}{\cos^2 x} \\
&= 1 + \sin x.
\end{aligned}$$

■

Formulating an appropriate procedure to verify an identity is sometimes elusive. When this is the case, a good "rule-of-thumb" is to convert the trigonometric expressions into a more familiar form, e.g. converting the expression into one of sine and cosine.

Example 5.20 Verify the identity.

$$\cos\alpha + \sec\alpha\sin^2\alpha = \sec\alpha.$$

Solution

$$\begin{aligned}
\cos\alpha + \sec\alpha\sin^2\alpha &= \cos\alpha + \left(\frac{1}{\cos\alpha}\right)\sin^2\alpha \\
&= \cos\alpha + \frac{\sin^2\alpha}{\cos\alpha} \\
&= \frac{\cos^2\alpha}{\cos\alpha} + \frac{\sin^2\alpha}{\cos\alpha} \\
&= \frac{\cos^2\alpha + \sin^2\alpha}{\cos\alpha} \\
&= \frac{1}{\cos\alpha} \\
&= \sec\alpha.
\end{aligned}$$

■

5.7 Exercises

* Exercise 1

Find the quadrant or axis in which each point lies.

(a) $A(0,4)$

(b) $B\left(5,-\frac{30}{7}\right)$

(c) $C(3,1)$

(d) $D\left(-4,0\right)$

(e) $E(1,-1.5)$

(f) $F(-\pi,4)$

(g) $G(-1,\sqrt{3})$

(h) $H\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$

* Exercise 2

Determine the quadrant or axis in which each standard position angle lies.

(a) $45°$

(b) $\frac{4\pi}{3}$

(c) $300°$

(d) $\frac{7\pi}{6}$

(e) $150°$

(f) $\frac{\pi}{2}$

(g) $315°$

(h) π

** Exercise 3

Find the point on the unit circle corresponding to $\theta =$

(a) $45°$

(b) $\frac{\pi}{3}$

(c) $210°$

(d) 0

(e) $\frac{7\pi}{4}$

(f) $120°$

(g) π

(h) $30°$

(i) $\frac{3\pi}{4}$

(j) $270°$

** Exercise 4

Determine the degree measure of the standard position angle α corresponding to each point on the unit circle. Assume that $0 \leq \alpha < 360°$.

(a) $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

(b) $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$

(c) $(0,1)$

(d) $\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$

(e) $(0,-1)$

(f) $\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$

** Exercise 5

Find the radian measure of the standard position angle β corresponding to each point on the unit circle. Suppose $0 \leq \beta < 2\pi$.

(a) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

(b) $(1, 0)$

(c) $(-1, 0)$

(d) $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

(e) $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

(f) $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

* Exercise 6

Use the unit circle to find following. Some values are undefined.

(a) $\cos 45°$

(b) $\tan 30°$

(c) $\tan 90°$

(d) $\sec 180°$

(e) $\sin 270°$

(f) $\csc 60°$

(g) $\cot 90°$

(h) $\sin 0$

* Exercise 7

Use the unit circle to find the following. Some values are undefined.

(a) $\sin \frac{\pi}{3}$

(b) $\tan \frac{3\pi}{2}$

(c) $\cos \frac{\pi}{3}$

(d) $\sec \frac{\pi}{4}$

(e) $\tan \pi$

(f) $\sec \frac{\pi}{6}$

(g) $\csc \frac{\pi}{3}$

(h) $\cot \pi$

** Exercise 8

Solve for θ where $0 \leq \theta < 360°$.

(a) $\sin \theta = 1$

(b) $\cos \theta + 1 = 0$

(c) $\tan \theta - 1 = 0$

(d) $\cos \theta = \frac{\sqrt{3}}{2}$

(e) $\csc \theta = \frac{2\sqrt{3}}{3}$

(f) $\tan \theta = -\sqrt{3}$

** Exercise 9

Solve for φ where $0 \leq \varphi < 2\pi$.

(a) $\cos \varphi = 0$

(b) $2 \sin \varphi - 1 = 0$

(c) $3 \tan \varphi = -\sqrt{3}$

(d) $\frac{1}{2} \sec \varphi = 1$

(e) $\cot \varphi + 1 = 2$

(f) $3 \csc \varphi = -6$

** Exercise 10

Determine the quadrant in which the terminal side of θ lies.

(a) $\sin\theta > 0$ and $\cos\theta > 0$

(b) $\cos\theta > 0$ and $\tan\theta < 0$

(c) $\sin\theta < 0$ and $\sec\theta < 0$

(d) $\csc\theta > 0$ and $\sec\theta < 0$

(e) $\sin\theta < 0$ and $\cot\theta < 0$

(f) $\sec\theta < 0$ and $\cot\theta > 0$

** Exercise 11

State whether the terminal side of φ lies in the positive x-axis, positive y-axis, negative x-axis, or negative y-axis.

(a) $\csc\varphi$ is undefined and $\cos\varphi > 0$.

(b) $\tan\varphi$ is undefined and $\csc\varphi > 0$.

(c) $\cot\varphi$ is undefined and $\sec\varphi < 0$.

(d) $\sec\varphi$ is undefined and $\sin\varphi < 0$.

** Exercise 12

Compute the reference angle.

(a) $147°$

(b) $314°$

(c) $29°$

(d) $307°$

(e) $217°$

(f) $201°$

(g) $316°$

(h) $118°$

** Exercise 13

Find the reference angle.

(a) $\frac{5\pi}{6}$

(b) $\frac{11\pi}{12}$

(c) $\frac{19\pi}{10}$

(d) $\frac{3\pi}{4}$

(e) $\frac{9\pi}{8}$

(f) $\frac{11\pi}{6}$

(g) $\frac{4\pi}{3}$

(h) $\frac{13\pi}{9}$

(i) 5

(j) 1

** Exercise 14

Evaluate.

(a) $\sin 30°$

(b) $\sin 150°$

(c) $\sin 210°$

(d) $\sin 330°$

** Exercise 15

Evaluate.

(a) $\cos\frac{\pi}{4}$

(b) $\cos\frac{3\pi}{4}$

(c) $\cos\frac{5\pi}{4}$

(d) $\cos\frac{7\pi}{4}$

** Exercise 16

Evaluate.

(a) $\tan 60°$

(b) $\tan 120°$

(c) $\tan 240°$

(d) $\tan 300°$

** Exercise 17

Use reference angles to evaluate. Some expressions are undefined.

(a) $\tan\frac{5\pi}{6}$

(b) $\cos\frac{3\pi}{2}$

(c) $\sin\frac{11\pi}{6}$

(d) $\tan\frac{\pi}{2}$

(e) $\csc\frac{7\pi}{4}$

(f) $\cot\frac{2\pi}{3}$

(g) $\csc\frac{7\pi}{6}$

(h) $\cot\frac{3\pi}{4}$

(i) $\sin\frac{5\pi}{4}$

(j) $\cos\frac{2\pi}{3}$

** Exercise 18

Use reference angles to evaluate. Some expressions are undefined.

(a) $\sin 120°$

(b) $\tan 315°$

(c) $\csc 315°$

(d) $\cos 270°$

(e) $\csc 240°$

(f) $\sec 210°$

(g) $\cos 150°$

(h) $\tan 330°$

(i) $\sin 225°$

(j) $\cos 120°$

** Exercise 19

Suppose α is an acute angle of degree measure such that the terminal side of α intersects the unit circle at the point $(4/5, 3/5)$.

(a) Find the values of the six trigonometric functions at α.

(b) What are the values of the six trigonometric functions at $360° - \alpha$?

(c) Compute the values of the six trigonometric functions at $\alpha + 180°$.

(d) Evaluate the six trigonometric functions at $180° - \alpha$.

** Exercise 20

Assume β is an acute angle of radian measure such that the terminal side of β intersects the unit circle at the point $(5/13, 12/13)$.

(a) Find the values of the six trigonometric functions at β.

(b) What are the values of the six trigonometric functions at $\pi - \beta$?

(c) Compute the values of the six trigonometric functions at $2\pi - \beta$.

(d) Evaluate the six trigonometric functions at $\beta + \pi$.

*** Exercise 21

The terminal side of θ lies in quadrant IV and intersects the unit circle at the point $(8/17, -15/17)$. Compute each of the following.

(a) $\sin\theta$

(b) $\cot\theta$

(c) $\cos(2\pi-\theta)$

(d) $\sec(\theta-\pi)$

(e) $\csc(\theta-\pi)$

(f) $\cos(3\pi-\theta)$

** Exercise 22

Compute each of the following. Some expressions are undefined.

(a) $\cot(-180°)$

(b) $\tan 870°$

(c) $\csc 585°$

(d) $\sin 630°$

(e) $\tan(-135°)$

(f) $\sin(-810°)$

(g) $\cot(-510°)$

(h) $\sec(-1050°)$

** Exercise 23

Calculate each of the following. Some expressions are undefined.

(a) $\cos 2\pi$

(b) $\cos\left(-\frac{9\pi}{2}\right)$

(c) $\csc\left(-\frac{11\pi}{3}\right)$

(d) $\sin\frac{7\pi}{2}$

(e) $\sin\frac{23\pi}{6}$

(f) $\csc\left(-\frac{13\pi}{4}\right)$

(g) $\sec\left(-\frac{5\pi}{6}\right)$

(h) $\cot\left(-\frac{9\pi}{4}\right)$

** Exercise 24

Prove Proposition 5.4 (v) and (vi).

** Exercise 25

The **domain** of a function is the set of inputs of a function, and the **range** of a function is the set of outputs of the function. Determine the domain and range of f.

(a) $f(x)=\sin x$

(b) $f(x)=\cos x$

(c) $f(x)=\tan x$

(d) $f(x)=\csc x$

(e) $f(x)=\sec x$

(f) $f(x)=\cot x$

** Exercise 26

Determine the values of θ which satisfy the equation.

(a) $\tan\theta=0$

(b) $-2\cos\theta=1$

(c) $3\cot^2\theta=1$

(d) $\sin^2\theta-2\sin\theta=-1$

** Exercise 27

Solve for φ.

(a) $2\sin 3\varphi=1$

(b) $-\tan 2\varphi=1$

(c) $3\csc^2\pi\varphi=4$

(d) $\sec^2(5\varphi)-3\sec(5\varphi)+2=0$

** Exercise 28

Find all values of α within the interval $[0, 360°)$ which satisfy the equation.

(a) $\tan(-\alpha) = \sqrt{3}$

(b) $3\csc 3\alpha = 3\sqrt{2}$

(c) $\cot^2 \frac{\alpha}{2} = 1$

(d) $2\sin^2(2\alpha) - 9\sin(2\alpha) = 5$

** Exercise 29

Solve for β. Assume $0 \le \beta < 2\pi$.

(a) $2\cos(-\beta) = \sqrt{3}$

(b) $\sec^2 \frac{\beta}{3} - 2 = 0$

(c) $\tan^2 \frac{\pi\beta}{3} = 3$

(d) $3\csc^2 \pi\beta - 5\csc \pi\beta + 2 = 0$

** Exercise 30

Find all values of θ in the interval $(-180°, 180°]$ which satisfy the equation.

(a) $6\sin(-\theta) = 3\sqrt{2}$

(b) $\csc^2 \frac{\theta}{2} - 2 = 0$

(c) $\cot \frac{5\theta}{4} + \sqrt{3} = 0$

(d) $\tan^2 \frac{3\theta}{2} - 1 = 0$

** Exercise 31

Determine the values of the remaining five trigonometric functions.

(a) $\sin\alpha = -4/5$ and $\cos\alpha < 0$

(b) $\cot\beta = 15/8$ and $\sec\beta > 0$

(c) $\cos\gamma = 5/13$ and $\cot\gamma < 0$

(d) $\tan\theta = -24/7$ and $\sec\theta < 0$

(e) $\csc\varphi$ is undefined and $\sec\varphi < 0$

** Exercise 32

Solve for α.

(a) $2\cos^2\alpha - \sin\alpha = 1$

(b) $3\tan^2\alpha = 4\sec\alpha + 1$

(c) $5\csc^4\alpha - 9\cot^2\alpha - 11 = 0$

** Exercise 33

Find all values of β.

(a) $\cos^4(3\beta) + 3\sin^2(3\beta) = 1$

(b) $\tan^3(2\beta) + \sec^2(-2\beta) = 3\tan(2\beta) + 4$

(c) $\csc^2(5\beta)\cot(-5\beta) = \csc^2(5\beta)$

** Exercise 34

(i) 1

(ii) $\tan x$

(iii) $-\cot x$

(iv) $\sec x$

Match the expressions above with the equivalent expressions below. Some options may be used more than once.

(a) $\dfrac{\sec^2 x - 1}{\tan x}$

(b) $\sin(x)\sec(-x)$

(c) $\cos(-x)\csc(-x)$

(d) $\cos x \sec x$

(e) $\sin(-x)\csc(-x)$

(f) $\dfrac{\tan x}{\sin x}$

** Exercise 35

Verify each identity.

(a) $\sin(2\pi - \alpha) = -\sin\alpha$

(b) $\cos(360^\circ - \beta) = \cos\beta$

(c) $\tan(2\pi - \gamma) = -\tan\gamma$

** Exercise 36

Verify the identity.

$$\frac{\tan(\theta + \pi)}{\sin\theta} = \sec\theta$$

** Exercise 37

Verify each identity.

(a) $\cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1$

(b) $\cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha$

** Exercise 38

Verify each identity.

(a) $\dfrac{\sin^2\theta}{1+\cos\theta} = 1 - \cos\theta$

(b) $\dfrac{\cos^2\alpha}{1+\sin\alpha} = 1 - \sin\alpha$

(c) $\dfrac{\sec^2\varphi - 1}{\tan\varphi} = \tan\varphi$

(d) $\frac{\tan\beta}{1-\cos\beta} = \sec\beta\csc\beta + \csc\beta$

** Exercise 39

Verify each identity.

(a) $\frac{\cos^2 x - \sin x(-\cos x)}{\cos^2 x} = \sec^2 x$

(b) $\frac{-\sin^2(-x) - \cos^2 x}{\sin^2 x} = -\csc^2 x$

** Exercise 40

Verify each identity.

(a) $\sec x + \tan x = \dfrac{1}{\sec x - \tan x}$

(b) $\csc x + \cot x = \dfrac{1}{\csc x - \cot x}$

** Exercise 41

Verify the identity.

$$\sin\theta + \sin\theta\tan^2\theta = \sec\theta\tan\theta$$

Chapter 6

Graphing Trigonometric Functions

In this chapter, we will learn to graph the six trigonometric functions on the xy-plane. We assume a thorough understanding of Chapter 5. Some knowledge of Appendix **??**—which addresses shifts, stretches, and compressions—is helpful, but will not be directly utilized. We will not use calculators.

6.1 Graphing Sine and Cosine

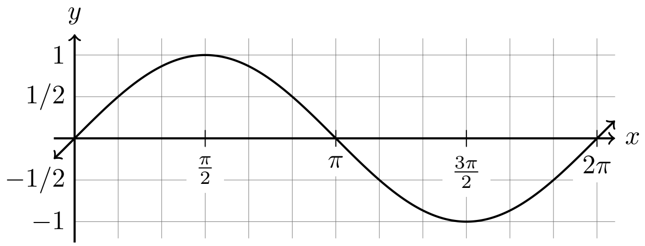
We want to graph functions of the form

$$f(x) = A\sin(Bx + C) + D \quad \text{and} \quad g(x) = A\cos(Bx + C) + D.$$

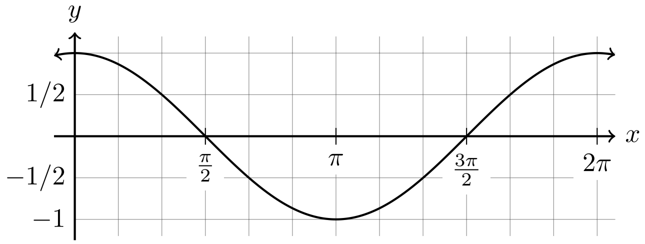
Definition 6.1 A **parent function** is considered to be the most basic within a family of functions.

We like to think of $f(x) = \sin x$ and $g(x) = \cos x$ as the parent functions of their respective family of functions. From this perspective, the constants A, B, C, and D modify the parent functions. As a result, it is helpful to obtain an understanding of the graphs of $f(x) = \sin x$ and $g(x) = \cos x$.

$f(x) = \sin x$



$g(x) = \cos x$



Proposition 5.3 tells us that sine and cosine are periodic with period 2π. This means that the behavior of $f(x) = \sin x$ and $g(x) = \cos x$, within the interval $[0, 2\pi]$, is repeated in subsequent intervals, so graphing more periods of either function is simply a matter of recognizing the pattern.

Definition 6.2

- The **amplitude** of f is

$$\frac{\max\{f(x)\} - \min\{f(x)\}}{2}.$$

- The **phase shift** of a periodic function f is how much its principal period is shifted left or right on a graph relative to the parent function of f.

Proposition 6.1 *Suppose we have a function of the form*

$$f(x) = A\sin(Bx + C) + D \quad \textit{or} \quad g(x) = A\cos(Bx + C) + D,$$

where $B > 0$*.*

(i) The amplitude of the function is $|A|$*.*

(ii) The graph has period

$$\frac{2\pi}{B}.$$

(iii) The neutral vertical position of its graph is $y = D$*; this is called the vertical shift.*

(iv) The phase shift is

$$-\frac{C}{B}.$$

Note: a positive phase shift corresponds to a rightward shift and a negative phase shift corresponds to a leftward shift.

Proposition 6.1 will be used extensively to graph functions, but plotting a small number of points is also helpful. We will introduce a protocol to find five useful points.

> Let $f(x) = A\sin(Bx + C) + D$ or $g(x) = A\cos(Bx + C) + D$, where $B > 0$.
>
> - Start at an x-coordinate equal to the value of the phase shift, i.e. start at
>
> $$x = -\frac{C}{B}.$$
>
> Evaluate to find the y-coordinate.
>
> - To find subsequent x-coordinates, add one-fourth the period to the previous x-coordinate. That is, add
>
> $$\frac{2\pi/B}{4} = \frac{\pi}{2B}$$
>
> to the previous x-coordinate. Evaluate the new x-value to find the corresponding y-value.

- Stop after you have found the point whose x-coordinate is equal to the phase shift plus the period. That is, stop after the point corresponding to

$$x = -\frac{C}{B} + \frac{2\pi}{B}.$$

This protocol gives five points. The y-coordinate of the last point should be the same as the first.

Example 6.1 Graph one period of

$$f(x) = 2\sin(3x - \pi) + 1.$$

Solution Proposition 6.1 gives us the table below.

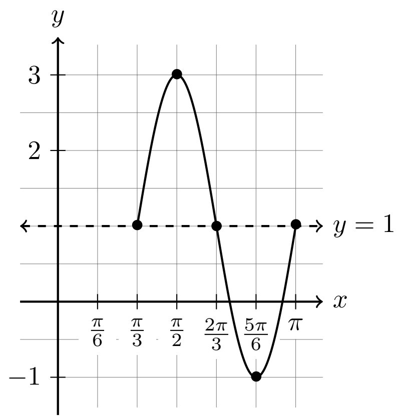
Amplitude:	$\lvert 2\rvert = 2$
Period:	$\frac{2\pi}{3}$
Vertical shift:	1
Phase shift:	$-\frac{(-\pi)}{3} = \frac{\pi}{3}$

Let us plot some points. The first x-coordinate is equal to the phase shift $\pi/3$ and we increase each subsequent x-coordinate by an increment of

$$\frac{2\pi/3}{4} = \frac{\pi}{6}.$$

x	$f(x)$
$\frac{\pi}{3}$	$2\sin(0)+1 = 1$
$\frac{3\pi}{6}=\frac{\pi}{2}$	$2\sin\left(\frac{\pi}{2}\right)+1 = 3$
$\frac{4\pi}{6}=\frac{2\pi}{3}$	$2\sin(\pi)+1 = 1$
$\frac{5\pi}{6}$	$2\sin\left(\frac{3\pi}{2}\right)+1 = -1$
π	$2\sin(2\pi)+1 = 1$

Hence, we have the following graph.



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Example 6.2 Graph two periods of

$$g(x) = -2 - \frac{1}{2}\cos\left(\frac{3\pi x + \pi}{4}\right).$$

Solution Let us rewrite this into a more familiar form:

$$g(x) = -2 - \frac{1}{2}\cos\left(\frac{3\pi x + \pi}{4}\right) = -\frac{1}{2}\cos\left(\frac{3\pi}{4}x + \frac{\pi}{4}\right) - 2.$$

Proposition 6.1 tells us the key features of g's graph.

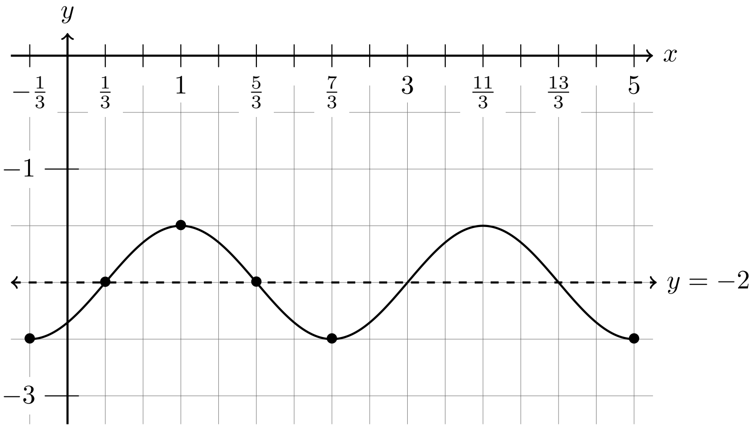
Amplitude:	$\left\lvert -\frac{1}{2} \right\rvert = \frac{1}{2}$
Period:	$\frac{2\pi}{3\pi/4} = \frac{8}{3}$
Vertical shift:	-2
Phase shift:	$-\frac{\pi/4}{3\pi/4} = -\frac{1}{3}$

Let us plot some points. The first point has an x-coordinate of $-1/3$ and subsequent x-values will be increased in increments of

$$\frac{8/3}{4} = \frac{2}{3}.$$

x	$g(x)$		
$-\frac{1}{3}$	$-\frac{1}{2}\cos(0) - 2$	$=$	$-2\frac{1}{2}$
$\frac{1}{3}$	$-\frac{1}{2}\cos\left(\frac{\pi}{2}\right) - 2$	$=$	-2
$\frac{3}{3} = 1$	$-\frac{1}{2}\cos(\pi) - 2$	$=$	$-1\frac{1}{2}$
$\frac{5}{3}$	$-\frac{1}{2}\cos\left(\frac{3\pi}{2}\right) - 2$	$=$	-2
$\frac{7}{3}$	$-\frac{1}{2}\cos(2\pi) - 2$	$=$	$-2\frac{1}{2}$

After we plot the points and graph the first period, we use the pattern to graph the second period.



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Proposition 6.1 assumes that $B > 0$. For $B < 0$, we can utilize Proposition 5.4 which says that sine is odd and cosine is even. This allows us to change the sign of the coefficient in front of x.

Example 6.3 Graph one period of

$$h(x) = 2\sin(-x + 45^\circ).$$

Solution Sine is odd, which implies

$$2\sin(-x + 45^\circ) = 2\sin\Big(-(x - 45^\circ)\Big) = -2\sin(x - 45^\circ).$$

Then we utilize Proposition 6.1.

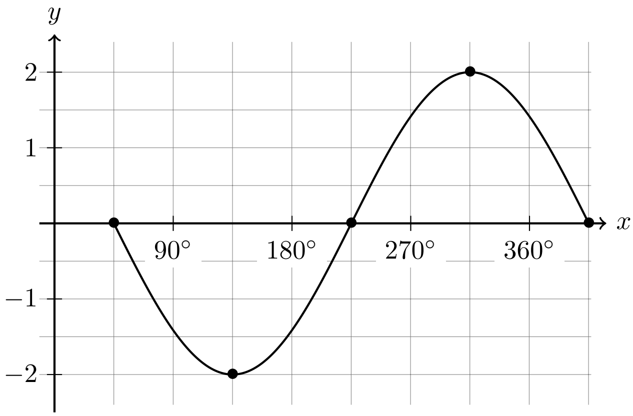
Amplitude:	$\lvert -2 \rvert = 2$
Period:	$\frac{360^\circ}{1} = 360^\circ$
Vertical shift:	0
Phase shift:	$-\frac{-45^\circ}{1} = 45^\circ$

The next step is to plot points. We begin at $x = 45^\circ$ and add

$$\frac{360^\circ}{4} = 90^\circ$$

to the previous x-coordinate to find the next. We stop at $x = 405^\circ$.

x	$h(x)$		
45°	$-2\sin 0$	$=$	0
135°	$-2\sin 90^\circ$	$=$	-2
225°	$-2\sin 180^\circ$	$=$	0
315°	$-2\sin 270^\circ$	$=$	2
405°	$-2\sin 360^\circ$	$=$	-2



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6.2 Graphing Tangent and Cotangent

The goal of this section is to graph functions of the form

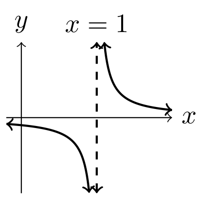
$$f(x) = A\tan(Bx + C) + D \quad \text{and} \quad g(x) = A\cot(Bx + C) + D.$$

Much like sine and cosine graphs, we think of A, B, C, and D as modifying the graphs of the parent functions $f(x) = \tan x$ and $g(x) = \cot x$.

Tangent and cotangent are somewhat more difficult to graph because they contain vertical asymptotes.

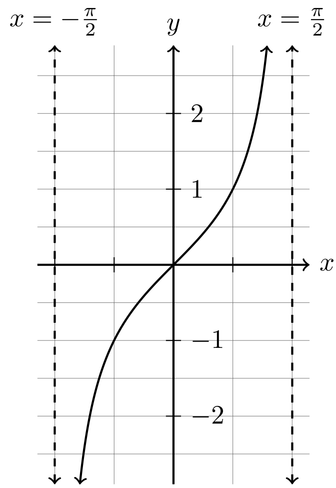
Definition 6.3 The function h has a **vertical asymptote** of $x = a$ if $h(x)$ goes to $\pm\infty$ as x goes to a from the left or the right. Within sketches of graphs, asymptotes are usually denoted by dashed lines.

The graph of h to the right has a vertical asymptote of $x = 1$. This is because $h(x)$ goes to $-\infty$ as x goes to 1 from the left, and $h(x)$ goes to ∞ as x goes to 1 from the right.



6.2.1 Graphing Tangent

Let us consider the graph of $f(x) = \tan x$.



Graphing more of $f(x) = \tan x$ is not difficult, because Proposition 5.3 tells us tangent has period π. Hence, in subsequent intervals, the graph of tangent simply repeats its behavior.

Proposition 6.2 *Suppose*

$$f(x) = A \tan(Bx + C) + D,$$

where $B > 0$.

(i) The graph has period

$$\frac{\pi}{B}.$$

(ii) The neutral vertical position of the graph is $y = D$*; this is called the vertical shift.*

(iii) The phase shift of f *is*

$$-\frac{C}{B}.$$

Note: Being shifted a negative number of units right corresponds to being shifted left.

(iv) The function f has vertical asymptotes at the solutions of

$$Bx + C = -\frac{\pi}{2} \quad \textit{and} \quad Bx + C = \frac{\pi}{2}.$$

Notice that we did not mentioned amplitude. The value of $|A|$ vertically compresses or stretches the graph of tangent, but it has no maximum or minimum value. This makes the concept of amplitude nonsensical.

Example 6.4 Determine the period, vertical shift, phase shift, and asymptotes of the function.

$$f(x) = -3\tan(15°x + 45°) - 7$$

Solution From Proposition 6.2, we know the period is

$$\frac{180°}{15°} = 12,$$

the vertical shift is -7, and the phase shift is

$$-\frac{45}{15} = -3.$$

One of the vertical asymptotes is the solution of

$$15°x + 45° = -90°.$$

Solving yields the vertical asymptote $x = -9$.

Since f has period 12, this means that all the asymptotes are of the form

$$x = -9 + 12n,$$

where $n = 0, 1, -1, 2, -2, \ldots$. ■

Proposition 6.2 will be used extensively, but we will also plot some points when we graph. So, we will introduce a protocol to find three helpful points.

> Suppose $f(x) = A\tan(Bx + C) + D$, where $B > 0$. Proposition 6.2 tells us that there are vertical asymptotes at
>
> $$Bx + C = -\frac{\pi}{2} \quad \text{and} \quad Bx + C = \frac{\pi}{2}.$$

- Add one-fourth the period to the x-value of the left asymptote. That is, add

$$\frac{\pi}{4B}$$

to the solution of $Bx + C = -\pi/2$. This gives the x-coordinate of the first point. Evaluate f at the x-value to find the y-coordinate.

- To find the next x-coordinate, add one-fourth the period to the previous x-coordinate. Evaluate f at the x-value to find the y-coordinate.

- Stop before you reach the solution of

$$Bx + C = \frac{\pi}{2}.$$

This protocol should give three points. Their y-coordinates should be $-A + D$, D, and $A + D$, respectively.

Example 6.5 Graph one period of

$$g(x) = 2\tan\left(\frac{x}{2} + \frac{\pi}{3}\right) - 4.$$

Solution Let us compute the vertical asymptotes. Using Proposition 6.2, the vertical asymptotes are the solutions of

$$\frac{x}{2} + \frac{\pi}{3} = -\frac{\pi}{2} \quad \text{and} \quad \frac{x}{2} + \frac{\pi}{3} = \frac{\pi}{2}.$$

Hence, we have vertical asymptotes

$$x = -\frac{5\pi}{3} \quad \text{and} \quad x = \frac{\pi}{3}.$$

Using the above, and some other parts of Proposition 6.2 give us our table.

Vertical shift:	-4
Period:	$\dfrac{\pi}{1/2} = 2\pi$
Phase shift:	$-\dfrac{\pi/3}{1/2} = -\dfrac{2\pi}{3}$
Asymptotes:	$x = -\dfrac{5\pi}{3}$ and $x = \dfrac{\pi}{3}$

Let us plot some points. Consecutive x-coordinates will be increased in increments of

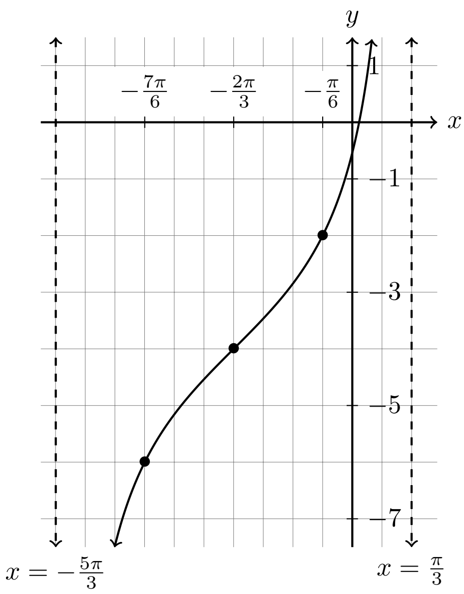
$$\frac{2\pi}{4} = \frac{\pi}{2}.$$

The first point has an x-coordinate of

$$-\frac{5\pi}{3} + \frac{\pi}{2} = -\frac{7\pi}{6}.$$

x	$g(x)$
$-\dfrac{7\pi}{6}$	$2\tan\left(-\dfrac{\pi}{4}\right) - 4 = -6$
$-\dfrac{4\pi}{6} = -\dfrac{2\pi}{3}$	$2\tan 0 - 4 = -4$
$-\dfrac{\pi}{6}$	$2\tan\left(\dfrac{\pi}{4}\right) - 4 = -2$

We have obtained enough information to graph one period of g.



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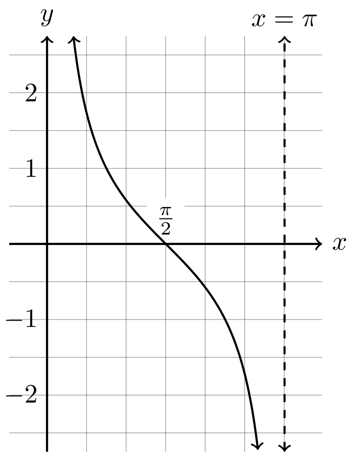
6.2.2 Graphing Cotangent

Consider the graph of $f(x) = \cot x$.



Proposition 5.3 tells us that cotangent has period π. Hence, the pattern within the interval $(0, \pi)$ is repeated in subsequent periods.

Proposition 6.3 *Suppose*

$$f(x) = A\cot(Bx + C) + D,$$

where $B > 0$.

(i) The graph has period

$$\frac{\pi}{B}.$$

(ii) The neutral vertical position of the graph is $y = D$*; this is called the vertical shift.*

(iii) The phase shift of f *is*

$$-\frac{C}{B}.$$

Note: being shifted a negative number of units right corresponds to being shifted left.

(iv) There are vertical asymptotes at

$$Bx + C = 0 \quad \textit{and} \quad Bx + C = \pi.$$

We will use some point plotting, along with Proposition 6.3, to graph cotangent. This makes a protocol for finding points necessary. Ours gives three useful points.

> Let $f(x) = A\cot(Bx + C) + D$, where $B > 0$. Proposition 6.3 tells us that f has vertical asymptotes at
>
> $$Bx + C = 0 \quad \text{and} \quad Bx + C = \pi.$$
>
> - Add one-fourth the period to the x-value of the left asymptote. That is, add
>
> $$\frac{\pi}{4B}$$
>
>   to the solution of $Bx + C = 0$. This gives the x-coordinate of the first point. Evaluate f at the x-value to find the y-coordinate.
>
> - To find the next x-coordinate, add one-fourth the period to the previous x-coordinate. Evaluate f at the x-value to find the y-coordinate.
>
> - Stop before you reach the x-value of the right asymptote. That is, stop before the x-value is the solution of
>
> $$Bx + C = \pi.$$
>
> This procedure gives three points. Their y-coordinates should be $A + D$, D, and $-A + D$, respectively.

Example 6.6 Graph two periods of

$$f(x) = \frac{1}{2}\cot(180^\circ x + 135^\circ) - 1.$$

Solution Using Proposition 6.3, the asymptotes within the first period are the solutions of

$$180^\circ x + 135^\circ = 0 \quad \text{and} \quad 180^\circ x + 135^\circ = 180^\circ.$$

Solving these yields

$$x = -\frac{3}{4} \quad \text{and} \quad x = \frac{1}{4}.$$

The period is

$$\frac{180^\circ}{180^\circ} = 1.$$

We need another asymptote, because we want to graph two periods of cotangent. Using the fact that cotangent has period 1, we conclude

$$x = \frac{1}{4} + 1 = \frac{5}{4}$$

is another asymptote.

We have the following table.

Period:	$\dfrac{180^\circ}{180^\circ} = 1$
Vertical shift:	-1
Phase shift:	$-\dfrac{135^\circ}{180^\circ} = -\dfrac{3}{4}$
Asymptotes:	$x = -\dfrac{3}{4}$, $x = \dfrac{1}{4}$, and $x = \dfrac{5}{4}$

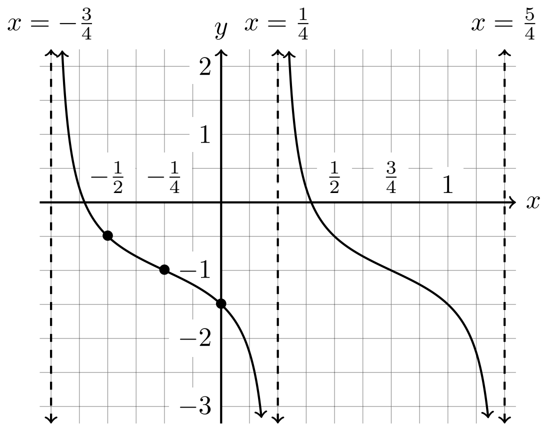
Next, we plot points within the first period. We increase subsequent x-coordinates by an increment $1/4$, and we start at

$$x = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}.$$

x	$f(x)$		
$-\frac{2}{4} = -\frac{1}{2}$	$\frac{1}{2}\cot(45^\circ) - 1$	$=$	$-\frac{1}{2}$
$-\frac{1}{4}$	$\frac{1}{2}\cot(90^\circ) - 1$	$=$	-1
0	$\frac{1}{2}\cot(135^\circ) - 1$	$=$	$-1\frac{1}{2}$

This enough to graph one period of f's graph. The next period is not difficult to draw, because we have the vertical asymptote $x =$

5/4 and we can determine the behavior of the graph by analyzing the previous period.



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Example 6.7 A cotangent graph has vertical asymptotes $x = 1$ and $x = 7$, no vertical shift, and contains the point $(5/2, 4)$. Find the equation of the corresponding cotangent function.

Solution Suppose

$$g(x) = A\cot(Bx + C) + D$$

is the function. There is no vertical shift which implies $D = 0$. Since $x = 1$ and $x = 7$ are vertical asymptotes, the period is

$$7 - 1 = 6.$$

Hence, Proposition 6.3 tells us

$$\frac{\pi}{B} = 6 \quad \text{implies} \quad B = \frac{\pi}{6}.$$

For cotangent functions, the phase shift is the same as the left asymptote within the principal period. So, it must be 1. Using Proposition 6.3 again, it follows that

$$-\frac{C}{\pi/6} = 1 \quad \text{implies} \quad C = -\frac{\pi}{6}.$$

As of now, we have

$$g(x) = A\cot\left(\frac{\pi}{6}x - \frac{\pi}{6}\right).$$

To find A, we will plug in $5/2$, because we know $g(5/2) = 4$:

$$\begin{aligned} g\left(\frac{5}{2}\right) &= A\cot\left(\frac{\pi}{6}\cdot\frac{5}{2} - \frac{\pi}{6}\right) \\ &= A\cot\frac{\pi}{4} \\ &= A(1) \\ &= A. \end{aligned}$$

We conclude $A = 4$.

Thus, our function is

$$g(x) = 4\cot\left(\frac{\pi}{6}x - \frac{\pi}{6}\right).$$

■

6.3 Graphing Secant and Cosecant

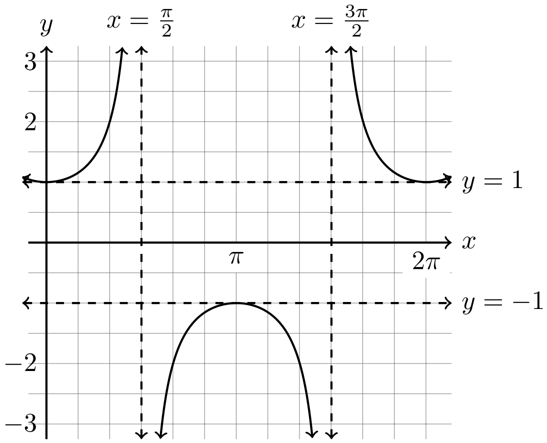
In this section, we will learn to graph secant and cosecant. These graphs rely on the skills from Section 6.1, so we recommend that the reader masters those concepts before they continue. In particular, because

$$\sec x = \frac{1}{\cos x} \quad \text{and} \quad \csc x = \frac{1}{\sin x},$$

the reader must understand how to graph cosine to graph secant, and how to graph sine to graph cosecant.

6.3.1 Graphing Secant

Let us begin with an analysis of the graph of $f(x) = \sec x$.



Each branch of secant is a U-shaped curve. Secant has asymptotes at $x = \pi/2$ and $x = 3\pi/2$. From Proposition 5.3, we know that secant has period 2π. As a result, we can graph more periods easily.

We now introduce a procedure to graph secant.

Let

$$f(x) = A \sec(Bx + C) + D.$$

1. Lightly sketch the graph of

$$y = A \cos(Bx + C) + D.$$

2. Draw vertical asymptotes where the cosine graph intersects the line $y = D$.

3. Draw horizontal dashed lines $y = A + D$ and $y = -A + D$. These values correspond to the maximum and minimum y-values of $y = A\cos(Bx + C) + D$.

4. Draw the graph of

$$f(x) = A \sec(Bx + C) + D.$$

   Each U- shaped branch touches the cosine graph at its vertex and opens away from the cosine graph.

This strategy is based on the fact that secant is the reciprocal function of cosine.

Example 6.8 Graph one period of

$$f(x) = -2 - \frac{1}{2}\sec\left(\frac{3\pi x + \pi}{4}\right).$$

Solution The first step is to graph

$$y = -2 - \frac{1}{2}\cos\left(\frac{3\pi x + \pi}{4}\right) = -\frac{1}{2}\cos\left(\frac{3\pi}{4}x + \frac{\pi}{4}\right) - 2.$$

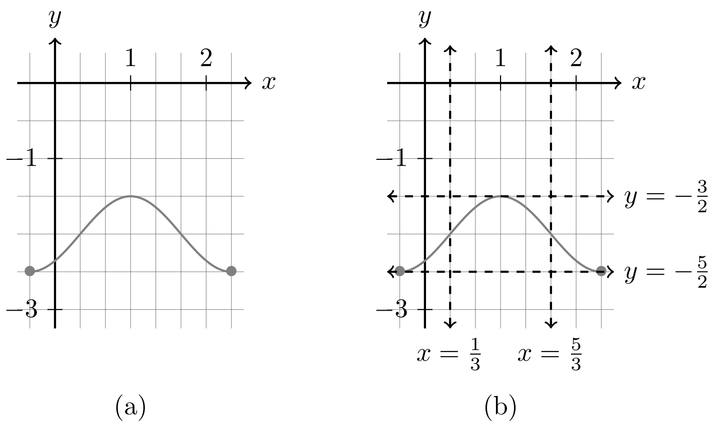
Using Section 6.1, and in particular Example 2, yields (a). Then we draw horizontal dashed lines at

$$y = -\frac{1}{2} - 2 = -\frac{5}{2} \quad \text{and} \quad y = \frac{1}{2} - 2 = -\frac{3}{2}.$$

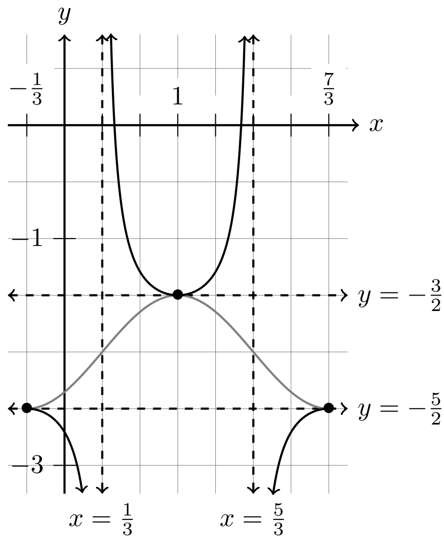
Vertical asymptotes are drawn at

$$x = \frac{1}{3} \quad \text{and} \quad x = \frac{5}{3},$$

because this is where the cosine graph intersects $y = -2$. This gives (b).



All that is left is to draw secant's branches. The branches have vertices of $(-1/3, -5/2)$, $(1, -3/2)$, and $(-7/3, -5/2)$, because those are the points where cosine intersects the dashed lines. The rest of each U-shape follows due to the position of the asymptotes.

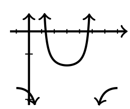


■

Note that only the darkened curves above is the secant graph; the rest is a graphing aid. So, plugging

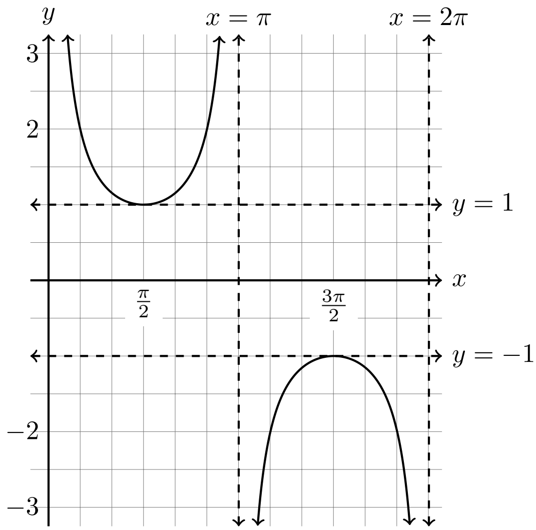
$$f(x) = -2 - \frac{1}{2}\sec\left(\frac{3\pi x + \pi}{4}\right)$$

into a graphing calculator results in a graph like the one on the right.



6.3.2 Graphing Cosecant

Consider the graph of $f(x) = \csc x$.



Notice that the principal period of cosecant has two U-shaped branches. Its vertical asymptotes occur at $x = 0$, $x = \pi$, and $x = 2\pi$. Graphing more branches of cosecant is a matter of emulating cosecant's behavior in the interval $(0, 2\pi)$. This is because of Proposition 5.3 which says cosecant's period is 2π.

Our procedure to graph

$$f(x) = A\csc(Bx + C) + D$$

is nearly identical to secant's which is described on page 84. Simply replace $y = A\cos(Bx + C) + D$ with $y = A\sin(Bx + C) + D$, and draw the U-shaped branches based on the latter's graph.

Example 6.9 Graph two periods of

$$g(x) = 2\csc(3x - \pi) + 1.$$

Solution In Example 1 of Section 6.1, we graphed

$$y = 2\sin(3x - \pi) + 1$$

for $\pi/3 \leq x \leq \pi$, which was one period.

We need two periods of the sine graph to obtain two periods of cosecant's graph. To keep the graph close to the y-axis, we will graph $y = 2\sin(3x - \pi) + 1$ for the period corresponding $-\pi/3 \leq x \leq \pi/3$. The result is (a).

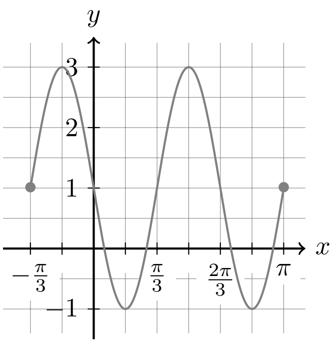
We draw dashed horizontal lines at

$$y = 2 + 1 = 3 \quad \text{and} \quad y = -2 + 1 = -1.$$

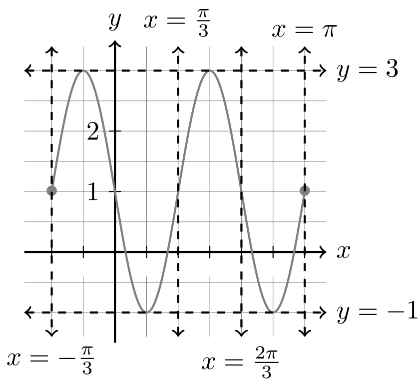
Vertical asymptotes are drawn at

$$x = -\frac{\pi}{3}, \quad x = 0, \quad x = \frac{\pi}{3}, \quad x = \frac{2\pi}{3}, \quad \text{and} \quad x = \pi,$$

because this is where $y = 2\sin(3x - \pi) + 1$ intersects $y = 1$. The result is (b).

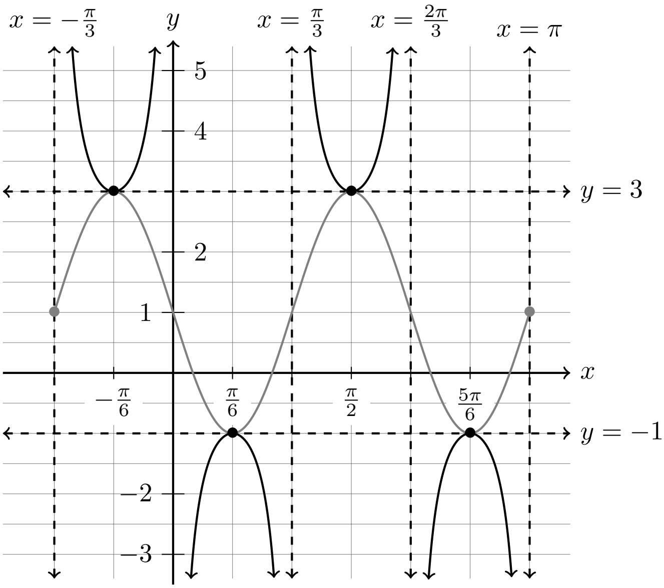


(a)



(b)

We are ready to graph cosecant. The vertices for the branches are at $(-\pi/6, 3)$, $(\pi/6, -1)$, $(\pi/2, 3)$, and $(5\pi/6, -1)$. The U-shaped branches follow from the asymptotes.



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6.4 Miscellaneous Graphing Problems

In this section, we will examine some less essential graphing problems. We will study them for fun as well as the fact that working a few challenging problems is important for students' mathematical development.

Example 6.10 Graph

$$f(x) = x \sin x.$$

Solution Since

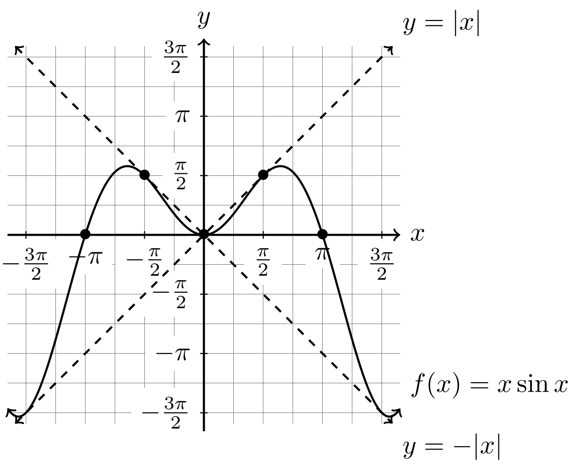
$$-1 \leq \sin x \leq 1 \quad \text{implies} \quad -|x| \leq x \sin x \leq |x|,$$

the graph of f oscillates between $y = -|x|$ and $y = |x|$.

Plotting points for x in the interval $[-\pi, \pi]$ is helpful because sine has period 2π and the sign change of x at $x = 0$ affects the look of the graph. We can then use the behavior we observe from our points to graph more of f.

Our first point will have an x-coordinate of $-\pi$ and subsequent x-coordinates will be $2\pi/4 = \pi/2$ greater than their previous x-coordinate.

x	$f(x)$
$-\pi$	$-\pi \sin(-\pi) = 0$
$-\frac{\pi}{2}$	$-\frac{\pi}{2} \sin\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$
0	$0 \sin 0 = 0$
$\frac{\pi}{2}$	$\frac{\pi}{2} \sin \frac{\pi}{2} = \frac{\pi}{2}$
π	$\pi \sin \pi = 0$



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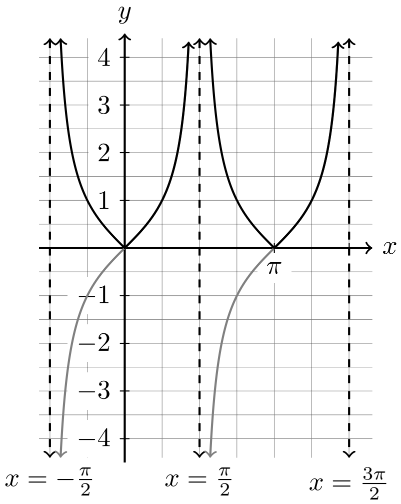
Example 6.11 Graph two periods of

$$g(x) = |\tan x|\,.$$

Solution We will use the graph of $y = \tan x$ as an aid. When $\tan x \geq 0$, the absolute value does nothing so we leave the graph of $y = \tan x$ unaltered. When $\tan x < 0$, the absolute value makes $g(x)$ positive so we reflect the graph of $y = \tan x$ about the x-axis.

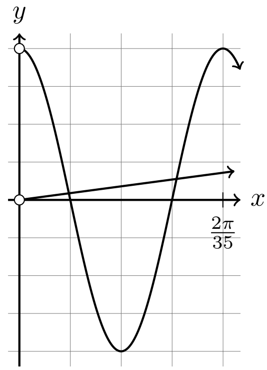
The graph of $g(x) = |\tan x|$ is drawn in black, and the graph of $y = \tan x$ is drawn in gray.



■

Example 6.12 Determine the number of times $y = x$ intersects $y = \cos 35x$ for $x > 0$.

Solution Let us examine one period, and make an inference about the general pattern.



During the first quarter of each period cosine goes from 1 to 0, and in the fourth quarter cosine goes from 0 to 1. When $0 \leq x \leq 1$, this implies that the graphs of $y = \cos 35x$ and $y = x$ intersect once in the first and fourth quarter of each period. When $x > 1$, $y = x$ will never intersect $y = \cos 35x$, because $-1 \leq \cos 35x \leq 1$.

As a result, we can find the number of positive intersects by computing the number of periods of $y = \cos 35x$ within the interval $[0, 1]$. The period of $y = \cos 35x$ is $2\pi/35$. It follows that there are

$$\frac{1}{2\pi/35} = \frac{35}{2\pi} \approx 5.570$$

periods between $x = 0$ and $x = 1$. Since $5.25 < 5.570 < 5.75$, $y = \cos 35x$ and $y = x$ intersect

$$2(5) + 1 = 11$$

times. ■

6.5 Exercises

* Exercise 1

Determine the amplitude, period, vertical shift, and phase shift.

(a) $y = 3\sin \pi x$

(b) $y = -\frac{1}{2}\cos(x - 30°) + 1$

(c) $y = -2\sin(2\pi(x-1)) - \pi$

(d) $y = 1 - \cos(x - \pi)$

(e) $y = -2\sin\left(\frac{x}{2} + \pi\right)$

(f) $y = \dfrac{3\cos\left(\frac{x+\pi}{3}\right) - 7}{4}$

** Exercise 2

Write a corresponding function.

(a) A sine function begins each period by decreasing from its neutral position.

Amplitude:	3
Period:	6
Vertical shift:	0
Phase shift:	2

(b) A cosine function begins each period at its maximum.

Amplitude:	π
Period:	$\pi/2$
Vertical shift:	-2
Phase shift:	$-\pi/6$

** Exercise 3

Graph one period.

(a) $y = 2\sin \pi x$

(b) $y = 1 - \cos(180°x - 30°)$.

(c) $y = -\frac{3}{4}\sin\left(\frac{\pi}{6}(x-2)\right) - 5$

(d) $y = -6\cos\left(\frac{x}{4} + \frac{\pi}{10}\right) + 1$

(e) $y = -\sin(90° + x) + 1$

(f) $y = 2\cos\left(\frac{\pi - x}{6}\right)$

(g) $y = \sin\left(\frac{\pi - 2x}{3}\right) + 1$

(h) $y = 7 - \cos(-\pi x)$

** Exercise 4

Graph two periods.

(a) $y = -3\sin 60°x$

(b) $y = 5 - \frac{1}{2}\cos 5x$

(c) $y = 2 - \sin\left(\frac{x - \pi}{3}\right)$

(d) $y = 6\cos\left(\frac{x + 18°}{5}\right)$

(e) $y = -\sin\left(-x + \frac{\pi}{4}\right) + 1$

(f) $y = -\frac{3}{4}\cos\left(\frac{\pi - x}{4}\right)$

** Exercise 5

Find A, B, and C such that ...

(a) ... $\cos x = A\sin(Bx + C)$.

(b) ... $\sin x = A\cos(Bx + C)$.

* Exercise 6

Determine the period, vertical shift, phase shift, and asymptotes.

(a) $y = -3\tan\left(\frac{x}{2} + 45°\right)$

(b) $y = \frac{3\pi}{2} - \cot(17°x)$

(c) $y = \frac{5\tan\left(\frac{x+\pi}{3}\right) + 2}{\pi}$

(d) $y = -3\cot(18°x + 15°) + 1$

(e) $y = \frac{3}{4} - \tan(x - \pi)$

(f) $y = -4\cot\left(\frac{\pi x - 3}{2}\right) + 1$

** Exercise 7

Write a corresponding function.

(a) A tangent function has vertical asymptotes of $x = -1$ and $x = 9$. Its vertical shift is 1, and it contains the point $(3/2, -2)$.

(b) A cotangent function with vertical asymptotes $x = 0$ and $x = 5\pi$. Its vertical shift is -3, and it contains the point $(5\pi/4, -7)$.

** Exercise 8

Graph one period.

(a) $y = -2\tan 60°x$

(b) $y = \frac{\pi}{2} - \pi\cot 2x$

(c) $y = -3\tan\left(\frac{\pi}{10}(x - 7)\right) - 5$

(d) $y = -3\cot\left(\frac{x}{5} - 20°\right) + 1$

(e) $y = -\tan(60° - x) + 1$

(f) $y = \pi\cot\left(\frac{\pi - x}{3}\right)$

(g) $y = \tan(15° - 3x) - 1$

(h) $y = -\cot\left(\frac{2\pi}{3}(2 - x)\right) + 2$

** Exercise 9

Graph two periods.

(a) $y = -\pi\tan\frac{\pi x}{6}$

(b) $y = 2 - \frac{3}{4}\cot 10°x$

(c) $y = 1 - \frac{1}{2}\tan\left(\frac{2x - \pi}{4}\right)$

(d) $y = -\cot(20°(x + 2)) + 3$

(e) $y = -\tan\left(-x + \frac{\pi}{8}\right) + 2$

(f) $y = \cot\left(\frac{\pi - x}{6}\right) + 1$

** Exercise 10

Find A, B, and C such that ...

(a) $\ldots \cot x = A\tan(Bx+C)$.

(b) $\ldots \tan x = A\cot(Bx+C)$.

* Exercise 11

Determine the period, vertical shift, phase shift, and asymptotes.

(a) $y = 2\sec(120°x) - 3$

(b) $y = -\pi\csc\left(x+\frac{\pi}{3}\right)+\frac{\pi}{6}$

(c) $y = 1 - \csc\left(4\left(x-\pi\right)\right)$

(d) $y = -2\sec\left(\frac{x}{2}+\pi\right)$

** Exercise 12

Write a corresponding function.

(a) A secant function has asymptotes $x = 1$, $x = 3$, and $x = 5$. Its vertical shift is 2. The vertex of a downward opening branch is $(2, -1)$.

(b) A cosecant function has asymptotes $x = \pi$, $x = 3\pi$, and $x = 5\pi$. Its vertical shift is -11π. The vertex of an upward opening branch is $(4\pi, \pi/2)$.

** Exercise 13

Graph one period.

(a) $y = -5\sec 9°x$

(b) $y = \csc\left(x+\frac{\pi}{2}\right)-5$

(c) $y = 2 - \sec\left(\frac{x-2\pi}{3}\right)$

(d) $y = \csc\left(\frac{\pi}{12}\left(x-4\right)\right)+1.$

(e) $y = 2\pi + \pi\sec\left(20°x+15°\right)$

(f) $y = -2\csc\left(\frac{x}{6}-\frac{\pi}{18}\right)+3$

(g) $y = -\sec\left(\frac{\pi}{4}-x\right)+1$

(h) $y = 3\csc\left(\frac{\pi+x}{4}\right)$

** Exercise 14

Graph two periods.

(a) $y = -2\sec\frac{2x}{3}$

(b) $y = \pi - \frac{\pi}{6}\csc 5x$

(c) $y = 3 - \sec\left(\frac{x+\pi}{8}\right)$

(d) $y = -\csc\left(\frac{\pi}{6}\left(2x+6\right)\right)+2.$

(e) $y = -2\sec\left(-x+\frac{\pi}{3}\right)$

(f) $y = -\frac{\pi}{8}\csc\left(\frac{3\pi-\pi x}{4}\right)$

** Exercise 15

Graph one period.

(a) $y = \dfrac{3\csc(x + 17^\circ) - 4}{5}$

(b) $y = 4\sin\left(\dfrac{x - \pi}{2}\right)$

(c) $y = \dfrac{3}{4}\tan\left(\dfrac{x - \pi}{3}\right)$

(d) $y = 3 - \cos 3x$

(e) $y = 1 - \cot\left(\frac{\pi}{4}(x - 3)\right)$.

(f) $y = 2\sec(2\pi(x - 1)) - 1$

** Exercise 16

Graph two periods.

(a) $y = 1 - \sin(20^\circ(x - 2))$

(b) $y = -4\csc\left(\dfrac{x + \pi}{6}\right)$

(c) $y = 3 - \tan\left(\dfrac{x - 2\pi}{3}\right)$

(d) $y = 1 - \sec(x - \pi)$

(e) $y = \dfrac{1}{2}\cot\left(\dfrac{x + 25^\circ}{10}\right)$

(f) $y = \cos(20^\circ x + 30^\circ) - 1$

** Exercise 17

Write corresponding equations for the graphs on pages 97 and 98.

*** Exercise 18

Graph each of the following.

(a) $y = x\cos x$

(b) $y = x\sin \pi x$

(c) $y = \cos x + x$

(d) $y = x\sec x$

*** Exercise 19

Graph each of the following.

(a) $y = |\cos x|$

(b) $y = |\cot x|$

(c) $y = |\csc x|$

*** Exercise 20

Suppose $x > 0$. Determine the number of times $y = x$ intersects each graph.

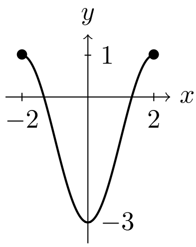
(a) $y = \sin 3x$

(b) $y = \cos 15x$

(c) $y = \sin 40x$

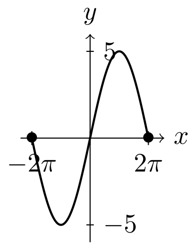
(d) $y = \cos 85x$

(e) $y = \tan x$



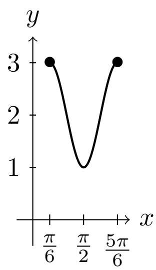
y
1
x
−2
2
−3

(a)



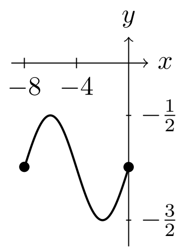
y
5
x
−2π
2π
−5

(b)



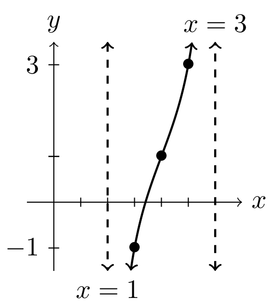
y
3
2
1
x
π/6
π/2
5π/6

(c)



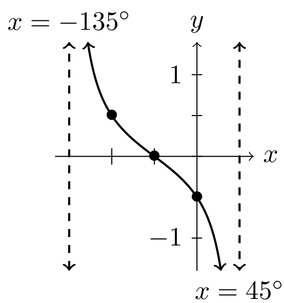
y
x
−8
−4
−1/2
−3/2

(d)



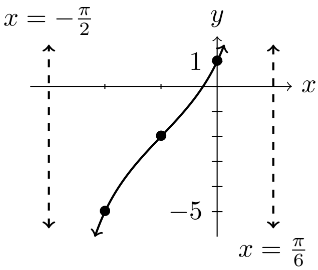
y
x = 3
3
x
−1
x = 1

(e)



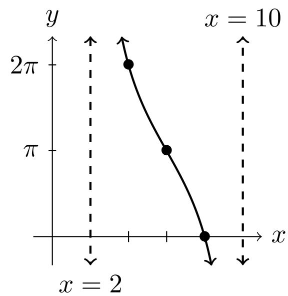
x = −135°
y
1
x
−1
x = 45°

(f)



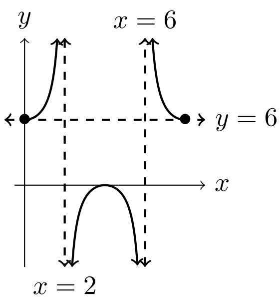
$x = -\frac{\pi}{2}$
y
1
x
-5
$x = \frac{\pi}{6}$

(g)



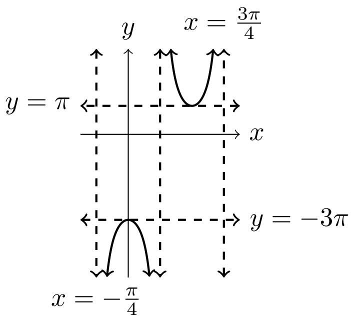
y
$x = 10$
2π
π
x
$x = 2$

(h)



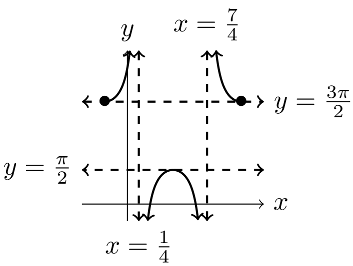
y
$x = 6$
$y = 6$
x
$x = 2$

(i)



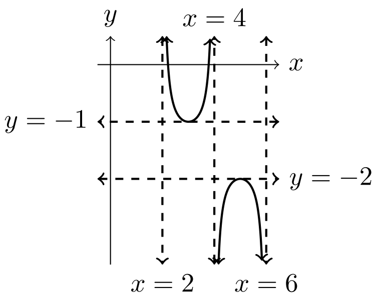
y
$x = \frac{3\pi}{4}$
$y = \pi$
x
$y = -3\pi$
$x = -\frac{\pi}{4}$

(j)



y
$x = \frac{7}{4}$
$y = \frac{3\pi}{2}$
$y = \frac{\pi}{2}$
x
$x = \frac{1}{4}$

(k)



y
$x = 4$
x
$y = -1$
$y = -2$
$x = 2$
$x = 6$

(l)

Chapter 7

Using Identities

In this chapter, we will examine some popular identities. We assume a solid command of Chapter 5. A modest amount of information from Chapter 6 will also be used. Calculators are not required. Indeed, many techniques discussed further expand the set of angles at which we can evaluate the trigonometric functions exactly by hand.

7.1 Sum and Difference Identities

Theorem 7.1 (Sum and Difference Identities) *Suppose α and β are standard position angles.*

(i) $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

(ii) $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

(iii) $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

The proof of Theorem 7.1 is difficult. As a result, it is broken up into pieces within this chapter. The proof of (ii) is in Subsection 7.1.1, (i) is proven in Subsection 7.2.1, and (iii) is left as an exercise for the reader.

Example 7.1 Evaluate

$$\sin\frac{\pi}{12}.$$

Solution Since

$$\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4},$$

Theorem 7.1 (i) tells us

$$\begin{aligned}\sin\frac{\pi}{12} &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\\ &= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2}\\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}.\end{aligned}$$

■

Example 7.2 Evaluate

$$\cos 195^\circ \cos 15^\circ + \sin 195^\circ \sin 15^\circ.$$

Solution Using Theorem 7.1 (ii),

$$\begin{aligned}\cos 195^\circ \cos 15^\circ + \sin 195^\circ \sin 15^\circ &= \cos(195^\circ - 15^\circ)\\ &= \cos 180^\circ\\ &= -1.\end{aligned}$$

■

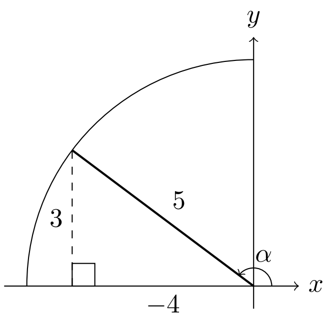
Example 7.3 Suppose the terminal side of α is in quadrant II and the terminal side of β is in quadrant III. Assume

$$\sin\alpha = \frac{3}{5} \quad\text{and}\quad \tan\beta = \frac{5}{12}.$$

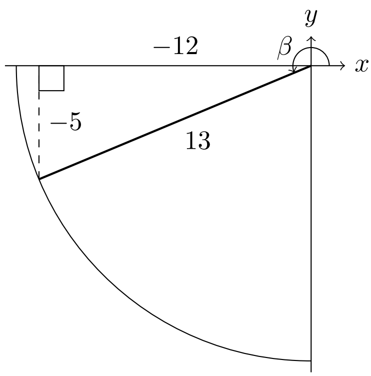
Compute (a) $\cos(\alpha+\beta)$ and (b) $\tan(\alpha-\beta)$.

Solution This will require Theorem 7.1 (ii) and (iii). However, in addition to the given information, the identities require $\cos\alpha$, $\tan\alpha$, $\sin\beta$, and $\cos\beta$. To find these values we will build triangles using the techniques outlined in Section 5.4.

Say the side opposite the reference angle of α is 3. Then the hypotenuse must be 5. The side adjacent the reference angle has signed length -4 due to the Pythagorean Theorem and the fact that the terminal side of α lies in quadrant II.



Let us say the side opposite β's reference angle has length 5. Then the side adjacent must have length 12. Because terminal side of β lies in quadrant III, these sides' signed lengths are -5 and -12, respectively. The Pythagorean Theorem tells us the length of the hypotenuse is 13.



We are ready to answer the questions.

(a)

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) \\ &= \frac{48}{65} + \frac{15}{65} \\ &= \frac{63}{65}.\end{aligned}$$

(b)

$$\begin{aligned}\tan(\alpha-\beta) &= \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} \\ &= \frac{-3/4 - 5/12}{1+(-3/4)(5/12)} \\ &= \frac{-14/12}{11/16} \\ &= -\frac{56}{33}.\end{aligned}$$

■

Example 7.4 Graph

$$y = 2\sqrt{3}\sin x - 2\cos x.$$

Solution Our goal is to use Theorem 7.1 (i) to rewrite the expression into the form

$$y = A\sin\left(Bx + C\right) + D,$$

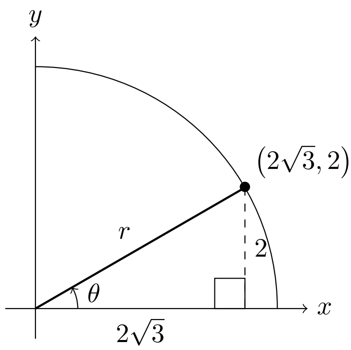
and then use the techniques discussed in Section 6.1 to graph the function. To do this, we will find a length r and a standard position angle θ such that

$$2\sqrt{3}\sin x - 2\cos x = r\left(\cos\theta\sin x - \sin\theta\cos x\right) = r\sin\left(x-\theta\right).$$

The terminal side of θ contains the point $\left(2\sqrt{3}, 2\right)$, because

$$r\cos\theta = 2\sqrt{3} \quad \text{and} \quad r\sin\theta = 2.$$

This allows us to build a triangle.



Using the Pythagorean Theorem,

$$\left(2\sqrt{3}\right)^2 + 2^2 = r^2 \quad \text{implies} \quad r = 4.$$

It follow that

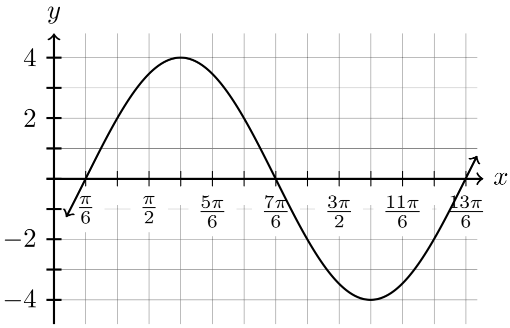
$$\cos\theta = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\theta = \frac{2}{4} = \frac{1}{2}.$$

From here, we see that $\theta = \pi/6$ satisfies the necessary criteria.

Then Theorem 7.1 (i) allows us to rewrite $y = 2\sqrt{3}\sin x - 2\cos x$:

$$\begin{aligned}
y &= 2\sqrt{3}\sin x - 2\cos x \\
&= 4\left(\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x\right) \\
&= 4\left(\cos\frac{\pi}{6}\sin x - \sin\frac{\pi}{6}\cos x\right) \\
&= 4\left(\sin x\cos\frac{\pi}{6} - \cos x\sin\frac{\pi}{6}\right) \\
&= 4\sin\left(x - \frac{\pi}{6}\right).
\end{aligned}$$

All that is left is to graph the result.



7.1.1 Proof of Theorem 7.1 (ii)

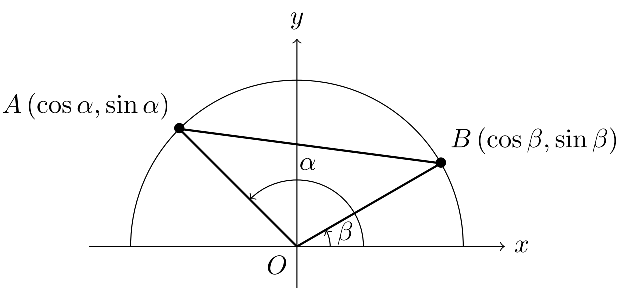
Theorem 7.1 (ii) says

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$$

Proof Suppose α and β are angles in standard position. Let A and B be the points

$$(\cos\alpha, \sin\alpha) \quad \text{and} \quad (\cos\beta, \sin\beta),$$

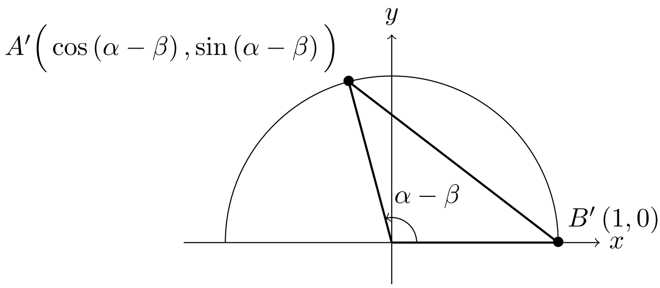
respectively.



Using the distance formula,

$$AB = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}.$$

Rotate $\triangle OBA$ measure β clockwise, so $\overline{OB}$ lies on the positive x-axis. Call the images of A and B under rotation A' and B' respectively. Then A' has coordinates $\Big(\cos(\alpha-\beta), \sin(\alpha-\beta)\Big)$ and B' has coordinates $(1,0)$.



Using the distance formula,

$$A'B' = \sqrt{\Big(\cos(\alpha-\beta)-1\Big)^2 + \sin^2(\alpha-\beta)}.$$

Rotations do not change lengths, so

$$A'B' = AB.$$

It follows that

$$\sqrt{\Big(\cos(\alpha-\beta)-1\Big)^2 + \sin^2(\alpha-\beta)} = \sqrt{\Big(\cos\alpha-\cos\beta\Big)^2 + (\sin\alpha-\sin\beta)^2}.$$

Squaring both sides yields

$$\Big(\cos(\alpha-\beta)-1\Big)^2 + \sin^2(\alpha-\beta) = \Big(\cos\alpha-\cos\beta\Big)^2 + \Big(\sin\alpha-\sin\beta\Big)^2.$$

Let us simplify each side separately. On the left side of the equation, we have

$$\begin{aligned}
&\Big(\cos(\alpha-\beta)-1\Big)^2 + \sin^2(\alpha-\beta) \\
&\qquad = \cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta) \\
&\qquad = \underbrace{\cos^2(\alpha-\beta) + \sin^2(\alpha-\beta)}_{1} - 2\cos(\alpha-\beta) + 1 \\
&\qquad = 2 - 2\cos(\alpha-\beta).
\end{aligned}$$

On the right side, we have

$$\begin{aligned}
&\Big(\cos\alpha - \cos\beta\Big)^2 + \Big(\sin\alpha - \sin\beta\Big)^2 \\
&\qquad = \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta \\
&\qquad = \underbrace{\cos^2\alpha + \sin^2\alpha}_{1} - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta + \underbrace{\cos^2\beta + \sin^2\beta}_{1} \\
&\qquad = 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta.
\end{aligned}$$

Hence,

$$\begin{aligned}
& 2 - 2\cos(\alpha-\beta) &&= 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta \\
\Rightarrow\quad & -2\cos(\alpha-\beta) &&= -2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta \\
\Rightarrow\quad & \cos(\alpha-\beta) &&= \cos\alpha\cos\beta + \sin\alpha\sin\beta
\end{aligned}$$

Now to prove

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Using Proposition 5.4, we know sine and cosine are odd and even, respectively. Thus,

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos\left(\alpha - (-\beta)\right) \\
&= \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta) \\
&= \cos\alpha\cos\beta + \sin\alpha\left(-\sin\beta\right) \\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta.
\end{aligned}$$

■

7.2 Other Identities

In this section, we will examine a few other popular trigonometric identities. Their proofs all follow from Theorem 7.1, albeit indirectly in some cases.

7.2.1 The Cofunction Identities

Proposition 7.1 (Cofunction Identities) *Let θ be a standard position angle.*

(i) $\sin(90^\circ - \theta) = \cos\theta$

(ii) $\cos(90^\circ - \theta) = \sin\theta$

(iii) $\tan(90^\circ - \theta) = \cot\theta$

(iv) $\cot(90^\circ - \theta) = \tan\theta$

(v) $\sec(90^\circ - \theta) = \csc\theta$

(vi) $\csc(90^\circ - \theta) = \sec\theta$

Proof The reader is given the opportunity to prove (i) in Exercise 12. We will prove (ii) and (iii).

(ii) Using Theorem 7.1 (ii),

$$\begin{aligned}\cos(90^\circ - \theta) &= \cos 90^\circ \cos\theta + \sin 90^\circ \sin\theta \\ &= 0 \cdot \cos\theta + 1 \cdot \sin\theta \\ &= \sin\theta.\end{aligned}$$

(iii) Assume (i) and (ii) hold. Then

$$\begin{aligned}\tan(90^\circ - \theta) &= \frac{\sin(90^\circ - \theta)}{\cos(90^\circ - \theta)} \\ &= \frac{\cos\theta}{\sin\theta} \\ &= \cot\theta.\end{aligned}$$

■

Example 7.5 Suppose $\tan 40^\circ \approx 0.839$. Without using a calculator, approximately what is the value of $\cot 50^\circ$?

Solution Since

$$\cot 50^\circ = \tan(90^\circ - 40^\circ) = \tan 40^\circ,$$

we conclude that

$$\cot 50^\circ \approx 0.839.$$

■

Example 7.6 Use the Cofunction Identities (i) and (ii) as well as Theorem 7.1 (ii) to prove

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha.$$

Solution Theorem 7.1 (ii) tells us

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta,$$

and Cofunction Identity (ii) says $\cos(90^\circ - \theta) = \sin\theta$. Therefore,

$$\begin{aligned}\sin(\alpha \pm \beta) &= \cos\Big(90^\circ - (\alpha \pm \beta)\Big)\\ &= \cos\Big((90^\circ - \alpha) \mp \beta\Big)\\ &= \cos(90^\circ - \alpha)\cos\beta \pm \sin(90^\circ - \alpha)\sin\beta\\ &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta.\end{aligned}$$

■

7.2.2 Double Angle Identities

Proposition 7.2 (Double Angle Identities) *Suppose θ is a standard position angle.*

(i) $\sin 2\theta = 2\sin\theta\cos\theta$

(ii) $\cos 2\theta = \begin{cases}\cos^2\theta - \sin^2\theta\\ 2\cos^2\theta - 1\\ 1 - 2\sin^2\theta\end{cases}$

(iii) $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

Proof The proofs for these properties are an application of Theorem 7.1.

(i) Using Theorem 7.1 (i),

$$\begin{aligned}\sin 2\theta &= \sin(\theta + \theta)\\ &= \sin\theta\cos\theta + \sin\theta\cos\theta\\ &= 2\sin\theta\cos\theta.\end{aligned}$$

(ii) Due to Theorem 7.1 (ii),

$$\begin{aligned}\cos 2\theta &= \cos(\theta+\theta)\\ &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ &= \cos^2\theta - \sin^2\theta.\end{aligned}$$

The other two variations of Proposition 7.2 (ii) following from the Pythagorean Identities

$$\sin^2\theta = 1-\cos^2\theta \quad \text{and} \quad \cos^2\theta = 1-\sin^2\theta.$$

We have

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= \cos^2\theta - (1-\cos^2\theta)\\ &= 2\cos^2\theta - 1\end{aligned}$$

and

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= (1-\sin^2\theta) - \sin^2\theta\\ &= 1-2\sin^2\theta.\end{aligned}$$

(iii) From Theorem 7.1 (iii),

$$\begin{aligned}\tan 2\theta &= \tan(\theta+\theta)\\ &= \frac{\tan\theta+\tan\theta}{1-\tan^2\theta}\\ &= \frac{2\tan\theta}{1-\tan^2\theta}.\end{aligned}$$

■

Example 7.7 Assume

$$\cos^2 2x = 2 - 5\cos x.$$

Solve for x.

Solution The second case of Proposition 7.2 (ii) tells us

$$\cos 2x = 2\cos^2 x - 1.$$

So,

$$\begin{aligned} && \cos^2 2x &= 2 - 5\cos x \\ \Rightarrow && 2\cos^2 x - 1 &= 2 - 5\cos x \\ \Rightarrow && 2\cos^2 x + 5\cos x - 3 &= 0 \\ \Rightarrow && (2\cos x - 1)(\cos x + 3) &= 0 \end{aligned}$$

It follows that

$$\cos x = \frac{1}{2} \quad \text{or} \quad \cos x = -3.$$

The latter case is impossible. If $\cos x = 1/2$, then

$$x = \frac{\pi}{3} + 2\pi n \quad \text{or} \quad x = \frac{5\pi}{3} + 2\pi n$$

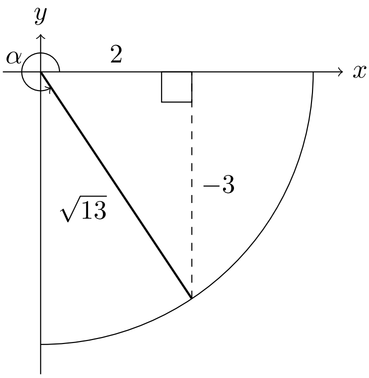
for $n = 0, 1, -1, 2, -2, \ldots$. ■

Example 7.8 Suppose

$$\tan\alpha = -\frac{3}{2} \quad \text{and} \quad \sec\alpha > 0.$$

Find (a) $\sin 2\alpha$, (b) $\cos 2\alpha$, and (c) $\tan 2\alpha$.

Solution The first step is to build a triangle using the techniques outlined in Section 5.4. Since $\tan\alpha < 0$ and $\sec\alpha > 0$, the terminal side of α is in quadrant IV. Say, the side opposite the reference angle has signed length of -3. Then the adjacent side has length 2. Due to the Pythagorean Theorem the hypotenuse has length $\sqrt{13}$.



(a) Using Proposition 7.2 (i),

$$\begin{aligned}\sin 2\alpha &= 2\sin\alpha\cos\alpha\\ &= 2\left(-\frac{3}{\sqrt{13}}\right)\left(\frac{2}{\sqrt{13}}\right)\\ &= -\frac{12}{13}.\end{aligned}$$

(b) Proposition 7.2 (ii) tells us

$$\begin{aligned}\cos 2\alpha &= 2\cos^2\alpha - 1\\ &= 2\left(\frac{2}{\sqrt{13}}\right)^2 - 1\\ &= -\frac{5}{13}.\end{aligned}$$

(c) Because of Proposition 7.2 (iii),

$$\begin{aligned}\tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha}\\ &= \frac{2(-3/2)}{1-(-3/2)^2}\\ &= \frac{12}{5}.\end{aligned}$$

■

The triangle we constructed in Example 8 was not required for (c). This is because we were given tangent.

7.2.3 Half Angle Identities

Proposition 7.3 (Half Angle Identities) *Assume θ is a standard position angle.*

(i) $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$

(ii) $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$

(iii) $\tan\frac{\theta}{2} = \begin{cases} \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\ \frac{1-\cos\theta}{\sin\theta} \\ \frac{\sin\theta}{1+\cos\theta} \end{cases}$

Proof

(i) Proposition 7.2 (ii) gives

$$\cos 2\alpha = 1 - 2\sin^2\alpha.$$

Solving for $\sin^2\alpha$ yields

$$\sin^2\alpha = \frac{1-\cos(2\alpha)}{2}.$$

Taking square roots and substituting $\theta/2$ for α gives

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}.$$

(ii) From Proposition 7.2 (ii),

$$\cos(2\alpha) = 2\cos^2\alpha - 1.$$

Solving for $\cos^2 \alpha$ gives

$$\cos^2 \alpha = \frac{1 + \cos(2\alpha)}{2}.$$

Then, after taking square roots and replacing α with $\theta/2$, we have

$$\cos \frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}.$$

(iii) The proof for (iii) follows from (i) and (ii):

$$\begin{aligned} \tan \frac{\theta}{2} &= \frac{\pm\sqrt{\frac{1-\cos\theta}{2}}}{\pm\sqrt{\frac{1+\cos\theta}{2}}} = \\ &= \pm\sqrt{\frac{1-\cos\theta}{2} \cdot \frac{2}{1+\cos\theta}} \\ &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}. \end{aligned}$$

To prove the second and third cases of (iii), we will use the identity

$$1 - \cos^2\theta = \sin^2\theta.$$

We have

$$\begin{aligned} \tan \frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\ &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta} \cdot \frac{1-\cos\theta}{1-\cos\theta}} \\ &= \pm\sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}} \\ &= \pm\sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}} \\ &= \pm\left|\frac{1-\cos\theta}{\sin\theta}\right| \end{aligned}$$

$$\begin{aligned} \tan \frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\ &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta}} \\ &= \pm\sqrt{\frac{1-\cos^2\theta}{(1-\cos\theta)^2}} \\ &= \pm\sqrt{\frac{\sin^2\theta}{(1-\cos\theta)^2}} \\ &= \pm\left|\frac{\sin\theta}{1-\cos\theta}\right| \end{aligned}$$

Then a careful analysis of signs reveals

$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta} \quad \text{and} \quad \tan\frac{\theta}{2} = \frac{\sin\theta}{1-\cos\theta}.$$

■

Example 7.9 Use the Half Angle Identities to compute (a) $\sin 157.5^\circ$, (b) $\cos 157.5^\circ$, and (c) $\tan 157.5^\circ$.

Solution

(a) Using Half Angle Identity (i),

$$\sin 157.5^\circ = \sin\left(\frac{1}{2}\cdot 315^\circ\right) = \pm\sqrt{\frac{1-\cos 315^\circ}{2}}.$$

Because 315° is in quadrant IV and its reference angle is 45°, we know

$$\cos 315^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2}.$$

It follows that

$$\begin{aligned}\sin 157.5^\circ &= \pm\sqrt{\frac{1-\cos 315^\circ}{2}}\\ &= \pm\sqrt{\frac{1-\sqrt{2}/2}{2}}\\ &= \pm\sqrt{\frac{2-\sqrt{2}}{4}}\\ &= \pm\frac{\sqrt{2-\sqrt{2}}}{2}.\end{aligned}$$

Since 157.5° is in quadrant II, sine is positive. Hence,

$$\sin 157.5^\circ = \frac{\sqrt{2-\sqrt{2}}}{4}.$$

(b) Using Half Angle Identity (ii),

$$\cos 157.5^\circ = \cos\left(\frac{1}{2}\cdot 315^\circ\right) = \pm\sqrt{\frac{1+\cos 315^\circ}{2}}.$$

From (a), $\cos 315° = \sqrt{2}/2$. It follows that

$$\begin{aligned}\cos 157.5° &= \pm\sqrt{\frac{1+\cos 315°}{2}} \\ &= \pm\sqrt{\frac{1+\sqrt{2}/2}{2}} \\ &= \pm\sqrt{\frac{2+\sqrt{2}}{4}} \\ &= \pm\frac{\sqrt{2+\sqrt{2}}}{2}.\end{aligned}$$

Since 157.5° is in the quadrant II, cosine is negative. Thus,

$$\cos 157.5° = -\frac{\sqrt{2+\sqrt{2}}}{2}.$$

(c) Using the second Half Angle Identity (iii),

$$\tan 157.5° = \tan\left(\frac{1}{2}\cdot 315°\right) = \frac{1-\cos 315°}{\sin 315°}.$$

All is left is to plug in the appropriate values for sine and cosine. From our previous work, we know $\cos 315° = \sqrt{2}/2$. Because 315° is in quadrant IV and its reference angle is 45°,

$$\sin 315° = -\sin 45° - \frac{\sqrt{2}}{2}.$$

Ergo,

$$\begin{aligned}\tan 157.5° &= \frac{1-\cos 315°}{\sin 315°} \\ &= \frac{1-\sqrt{2}/2}{-\sqrt{2}/2} \\ &= \frac{1-\sqrt{2}/2}{-\sqrt{2}/2}\cdot\frac{2\sqrt{2}}{2\sqrt{2}} \\ &= \frac{2\sqrt{2}-2}{-2} \\ &= 1-\sqrt{2}.\end{aligned}$$

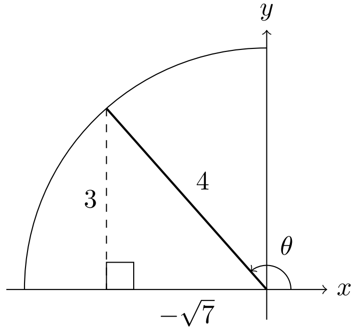
■

Example 7.10 Suppose θ is in $[0, 2\pi)$, and

$$\cot\theta = -\frac{\sqrt{7}}{3} \quad \text{and} \quad \cos\theta < 0.$$

What are the exact values of (a) $\sin(\theta/2)$, (b) $\cos(\theta/2)$, and (c) $\tan(\theta/2)$?

Solution The first step is to build a triangle. Since $\cot\theta < 0$ and $\cos\theta < 0$, the terminal side of θ lies in quadrant II. Suppose the signed length of the side adjacent the reference angle of θ is $-\sqrt{7}$. Then the side opposite has length 3. Using the Pythagorean Theorem, the hypotenuse must have length 4.



We are ready to answer the questions.

(a) Using Half Angle Identity (i),

$$\begin{aligned}\sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}}\\ &= \pm\sqrt{\frac{1-\left(-\sqrt{7}/4\right)}{2}}\\ &= \pm\sqrt{\frac{4+\sqrt{7}}{8}}\\ &= \pm\sqrt{\frac{8+2\sqrt{7}}{16}}\\ &= \pm\frac{\sqrt{8+2\sqrt{7}}}{4}.\end{aligned}$$

Since $\pi/2 < \theta < \pi$ implies $\pi/4 < \theta/2 < \pi/2$, we conclude $\sin(\theta/2)$ is positive. Thus,

$$\sin\frac{\theta}{2} = \frac{\sqrt{8+2\sqrt{7}}}{4}.$$

(b) Using Half Angle Identity (ii),

$$\begin{aligned}\cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}}\\ &= \pm\sqrt{\frac{1+\left(-\sqrt{7}/4\right)}{2}}\\ &= \pm\sqrt{\frac{4-\sqrt{7}}{8}}\\ &= \pm\sqrt{\frac{8-2\sqrt{7}}{16}}\\ &= \pm\frac{\sqrt{8-2\sqrt{7}}}{4}.\end{aligned}$$

Since $\pi/2 < \theta < \pi$ implies $\pi/4 < \theta/2 < \pi/2$, it follows that $\cos(\theta/2)$ is positive. Therefore,

$$\cos\frac{\theta}{2} = \frac{\sqrt{8-2\sqrt{7}}}{4}.$$

(c) Lastly, using the second case of Half Angle Identity (iii),

$$\begin{aligned}\tan\frac{\theta}{2} &= \frac{1-\cos\theta}{\sin\theta}\\ &= \frac{1+\sqrt{7}/4}{3/4}\\ &= \frac{4+\sqrt{7}}{3}.\end{aligned}$$

■

Another set of identities, which can be used to solve the same type of problems, are the Power Reducing Identities.

Corollary 7.1 (Power Reducing Identities) *Say that θ is a standard position angle.*

(i) $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

(ii) $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

(iii) $\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$

Readers that prefer these identities to the Half Angle Identities are welcome to use them instead.

7.2.4 Product to Sum and Difference Identities

Proposition 7.4 (Product to Sum and Difference Identities) *Suppose α and β are in $\mathbb{R}$.*

(i) $\sin\alpha\sin\beta = \dfrac{1}{2}\Big(\cos(\alpha-\beta) - \cos(\alpha+\beta)\Big)$

(ii) $\cos\alpha\cos\beta = \dfrac{1}{2}\Big(\cos(\alpha+\beta) + \cos(\alpha-\beta)\Big)$

(iii) $\sin\alpha\cos\beta = \dfrac{1}{2}\Big(\sin(\alpha+\beta) + \sin(\alpha-\beta)\Big)$

(iv) $\cos\alpha\sin\beta = \dfrac{1}{2}\Big(\sin(\alpha+\beta) - \sin(\alpha-\beta)\Big)$

These identities were invaluable for non-exact evaluation before students had access to calculators. In those days, students used tables to evaluate trigonometric expressions. As a result, these identities made it less cumbersome for students to evaluate products, because they were able to convert them into sums or differences which are easier to compute. Some programmers are still interested in the identities for the same reason.

There are other modern applications. The Product to Sum and Difference Identities allow students to find exact values of trigonometric functions at a slightly larger set of angles. The identities have applications in Calculus as well.

Proof

(i) Theorem 7.1 (ii) gives

$$\begin{array}{rrcrcr} & \cos\alpha\cos\beta & + & \sin\alpha\sin\beta & = & \cos(\alpha-\beta) \\ - & \Big(\cos\alpha\cos\beta & - & \sin\alpha\sin\beta & = & \cos(\alpha+\beta)\Big) \\ \hline & & & 2\sin\alpha\sin\beta & = & \cos(\alpha-\beta)-\cos(\alpha+\beta) \end{array}$$

Then dividing by 2 yields

$$\sin\alpha\sin\beta = \frac{1}{2}\Big(\cos(\alpha+\beta)-\cos(\alpha-\beta)\Big).$$

(ii) Using Theorem 7.1 (ii),

$$\begin{array}{rrcrcr} & \cos\alpha\cos\beta & - & \sin\alpha\sin\beta & = & \cos(\alpha+\beta) \\ + & \cos\alpha\cos\beta & + & \sin\alpha\sin\beta & = & \cos(\alpha-\beta) \\ \hline & 2\cos\alpha\cos\beta & & & = & \cos(\alpha+\beta)+\cos(\alpha-\beta). \end{array}$$

Then dividing by 2 yields

$$\cos\alpha\cos\beta = \frac{1}{2}\Big(\cos(\alpha+\beta)+\cos(\alpha-\beta)\Big).$$

(iii) From Theorem 7.1 (i),

$$\begin{array}{rrcrcr} & \sin\alpha\cos\beta & + & \cos\alpha\sin\beta & = & \sin(\alpha+\beta) \\ + & \sin\alpha\cos\beta & - & \cos\alpha\sin\beta & = & \sin(\alpha-\beta) \\ \hline & 2\sin\alpha\cos\beta & & & = & \sin(\alpha+\beta)+\sin(\alpha-\beta). \end{array}$$

Then dividing by 2 yields

$$\sin\alpha\cos\beta = \frac{1}{2}\Big(\sin(\alpha+\beta)+\sin(\alpha-\beta)\Big).$$

(iv) Because of Theorem 7.1 (i),

$$\begin{array}{rrcrcr} & \sin\alpha\cos\beta & + & \cos\alpha\sin\beta & = & \sin(\alpha+\beta) \\ - & \Big(\sin\alpha\cos\beta & - & \cos\alpha\sin\beta & = & \sin(\alpha-\beta)\Big) \\ \hline & & & 2\cos\alpha\sin\beta & = & \sin(\alpha+\beta)-\sin(\alpha-\beta). \end{array}$$

Then dividing by 2 yields

$$\cos\alpha\cos\beta = \frac{1}{2}\Big(\sin(\alpha+\beta) - \sin(\alpha-\beta)\Big).$$

■

Example 7.11 Compute (a) $\cos 45^\circ \cos 15^\circ$ and (b) $\sin 22.5^\circ \cos 22.5^\circ$.

Solution

(a) Using Proposition 7.4 (ii),

$$\begin{aligned}
\cos 45^\circ \cos 15^\circ &= \frac{1}{2}\Big(\cos(45^\circ + 15^\circ) + \cos(45^\circ - 15^\circ)\Big)\\
&= \frac{1}{2}\Big(\cos 60^\circ + \cos 30^\circ\Big)\\
&= \frac{1}{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)\\
&= \frac{1}{2}\left(\frac{1+\sqrt{3}}{2}\right)\\
&= \frac{1+\sqrt{3}}{4}.
\end{aligned}$$

(b) Due to Proposition 7.4 (iii),

$$\begin{aligned}
\sin 22.5^\circ \cos 22.5^\circ &= \frac{1}{2}\Big(\sin(22.5^\circ + 22.5^\circ) + \sin(22.5^\circ - 22.5^\circ)\Big)\\
&= \frac{1}{2}\Big(\sin 45^\circ + \sin 0\Big)\\
&= \frac{1}{2}\left(\frac{\sqrt{2}}{2} + 0\right)\\
&= \frac{\sqrt{2}}{4}.
\end{aligned}$$

■

7.3 Verifying Identities

We will verify identities in the final section of this chapter. The key ideas of the verification process were introduced in Section 5.6. However, this chapter has added more identities to our knowledge base, and the reader will be expected to utilize them.

Example 7.12 Verify

$$\frac{\csc(\pi/2-\alpha)}{1+\tan^2\alpha}=\cos\alpha.$$

Solution We need three identities: Pythagorean Identity (ii), Proposition 7.1 (iv), and a Reciprocal Identity. They say

$$1+\tan^2\alpha=\sec^2\alpha,\quad \csc\left(\frac{\pi}{2}-\alpha\right)=\sec\alpha,\quad \text{and}\quad \frac{1}{\sec\alpha}=\cos\alpha,$$

respectively. So,

$$\begin{aligned}\frac{\csc(\pi/2-\alpha)}{1+\tan^2\alpha}&=\frac{\csc(\pi/2-\alpha)}{\sec^2\alpha}\\&=\frac{\sec\alpha}{\sec^2\alpha}\\&=\frac{1}{\sec\alpha}\\&=\cos\alpha.\end{aligned}$$

■

Example 7.13 Verify that the equation is an identity.

$$\frac{\tan 2\beta}{\tan\beta} = \frac{2\cos^2\beta}{\cos^2\beta - \sin^2\beta}$$

Solution Recall that

$$\tan\beta = \frac{\sin\beta}{\cos\beta}.$$

We also need Proposition 7.2 (iii) which says

$$\tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta}.$$

Utilizing these identities, we have

$$\begin{aligned}
\frac{\tan 2\beta}{\tan\beta} &= \tan(2\beta)\frac{1}{\tan\beta} \\
&= \frac{2\tan\beta}{1-\tan^2\beta}\cdot\frac{1}{\tan\beta} \\
&= \frac{2}{1-\tan^2\beta} \\
&= \frac{2}{1-\frac{\sin^2\beta}{\cos^2\beta}} \\
&= \frac{2/1}{\frac{\cos^2\beta-\sin^2\beta}{\cos^2\beta}} \\
&= \frac{2}{1}\cdot\frac{\cos^2\beta}{\cos^2\beta-\sin^2\beta} \\
&= \frac{2\cos^2\beta}{\cos^2\beta-\sin^2\beta}.
\end{aligned}$$

■

Example 7.14 Verify the identity.

$$-\frac{1}{2}+\frac{1}{2}\sin\theta\cot\frac{\theta}{2}+\frac{1}{2}\cos\theta=\cos\theta.$$

Solution We need a Reciprocal Identity and the third case of Proposition 7.3 (iii), which say

$$\cot\frac{\theta}{2}=\frac{1}{\tan(\theta/2)}\quad\text{and}\quad\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}.$$

With our identities in mind, we proceed as follows:

$$\begin{aligned}-\frac{1}{2}+\frac{1}{2}\sin\theta\cot\frac{\theta}{2}+\frac{1}{2}\cos\theta &= -\frac{1}{2}+\frac{\sin\theta}{2}\cdot\frac{1}{\tan(\theta/2)}+\frac{1}{2}\cos\theta\\ &= -\frac{1}{2}+\frac{\sin\theta}{2}\cdot\frac{1+\cos\theta}{\sin\theta}+\frac{1}{2}\cos\theta\\ &= -\frac{1}{2}+\frac{1}{2}(1+\cos\theta)+\frac{1}{2}\cos\theta\\ &= -\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\cos\theta+\frac{1}{2}\cos\theta\\ &= \cos\theta.\end{aligned}$$

■

7.4 Exercises

** Exercise 1

Find the exact value.

(a) $\cos 285°$
(b) $\tan(-165°)$
(c) $\sin 375°$
(d) $\tan 255°$
(e) $\cos(-525°)$
(f) $\csc 345°$
(g) $\sec 105°$
(h) $\cot 15°$

** Exercise 2

Calculate the exact value.

(a) $\tan\left(-\frac{\pi}{12}\right)$
(b) $\sin \frac{19\pi}{12}$
(c) $\cot \frac{11\pi}{12}$
(d) $\sin\left(-\frac{35\pi}{12}\right)$
(e) $\sec \frac{23\pi}{12}$
(f) $\csc\left(-\frac{7\pi}{12}\right)$
(g) $\cos \frac{25\pi}{12}$
(h) $\cot\left(-\frac{85\pi}{12}\right)$

** Exercise 3

What is the exact value?

(a) $\cos 408° \cos 198° + \sin 408° \sin 198°$

(b) $\dfrac{\tan 57° + \tan 78°}{1 - \tan 57° \tan 78°}$

(c) $\sin 575° \cos 275° - \sin 275° \cos 575°$

(d) $\dfrac{\tan 312° - \tan 192°}{1 + \tan 312° \tan 197°}$

(e) $\cos 40° \cos 20° - \sin 40° \sin 20°$

(f) $\dfrac{\tan 286° - \tan 136°}{1 + \tan 286° \tan 136°}$

** Exercise 4

Compute the exact value.

(a) $\dfrac{\tan \frac{5\pi}{9} + \tan \frac{43\pi}{36}}{1 - \tan \frac{5\pi}{9} \tan \frac{43\pi}{36}}$

(b) $\sin \frac{19\pi}{9} \cos \frac{7\pi}{9} - \sin \frac{7\pi}{9} \cos \frac{19\pi}{9}$

(c) $\dfrac{\tan \frac{17\pi}{18} - \tan \frac{\pi}{9}}{1 + \tan \frac{17\pi}{18} \tan \frac{\pi}{9}}$

(d) $\cos \frac{5\pi}{24} \cos \frac{\pi}{24} - \sin \frac{5\pi}{24} \sin \frac{\pi}{24}$

(e) $\dfrac{\tan \frac{31\pi}{18} - \tan \frac{17\pi}{36}}{1 + \tan \frac{31\pi}{18} \tan \frac{17\pi}{36}}$

(f) $\cos \frac{\pi}{8} \cos \frac{23\pi}{24} + \sin \frac{\pi}{8} \sin \frac{23\pi}{24}$

** Exercise 5

Solve for θ.

(a) $\cos\left(\theta + \dfrac{\pi}{6}\right) = \sin\theta$

(b) $\sin\left(\theta + \frac{\pi}{4}\right) + \cos\left(\theta + \frac{\pi}{4}\right) = -1$

(c) $\tan\left(\theta + \dfrac{\pi}{6}\right) = -\sqrt{3}$

** Exercise 6

Suppose $\sin\alpha = 12/13$, $\tan\beta = -\sqrt{17}/8$, $\cos\alpha < 0$, and $\cos\beta > 0$.

(a) Find $\sin(\alpha + \beta)$.
(b) What is $\cos(\alpha - \beta)$?
(c) Evaluate $\tan(\alpha + \beta)$.
(d) Calculate $\csc(\alpha - \beta)$.

** Exercise 7

Assume $\cos\alpha = -4/5$, $\cot\beta = 15/8$, $\tan\alpha > 0$, and $\sin\beta > 0$.

(a) Find $\cos(\alpha - \beta)$.

(b) What is $\tan(\alpha - \beta)$?

(c) Evaluate $\sec(\alpha + \beta)$.

(d) Calculate $\cot(\alpha + \beta)$.

** Exercise 8

Write each expression in the form

$$y = A\sin(Bx + C) + D$$

for some A, B, C, and D.

(a) $y = 5\sin\pi x + 5\cos\pi x$

(b) $y = -3\sin x - 3\sqrt{3}\cos x$

(c) $y = \cos x - \sin x$

(d) $y = \sqrt{3}\sin 2x - \cos 2x$

** Exercise 9

Graph each expression.

(a) $y = 3\sqrt{2}\sin\frac{x}{2} + 3\sqrt{2}\cos\frac{x}{2}$

(b) $y = \sqrt{3}\sin x - \cos x$

(c) $y = -\frac{1}{4}\sin x - \frac{\sqrt{3}}{4}\cos x$

(d) $y = \sqrt{2}\cos\frac{\pi x}{2} - \sqrt{2}\sin\frac{\pi x}{2}$

* Exercise 10

$\sin 17^\circ \approx 0.292$	$\csc 43^\circ \approx 1.466$
$\cos 22^\circ \approx 0.927$	$\sec 55^\circ \approx 1.743$
$\tan 77^\circ \approx 4.331$	$\cot 25^\circ \approx 2.145$

Find the approximate value without a calculator.

(a) $\cot 13^\circ$

(b) $\cos 73^\circ$

(c) $\csc 35^\circ$

(d) $\tan 65^\circ$

(e) $\sin 68^\circ$

(f) $\sec 47^\circ$

* Exercise 11

$\sin\frac{5\pi}{16} \approx 0.831$	$\csc\frac{\pi}{7} \approx 2.305$
$\cos\frac{2\pi}{5} \approx 0.309$	$\sec\frac{5\pi}{18} \approx 1.556$
$\tan\frac{7\pi}{11} \approx -2.190$	$\cot\frac{11\pi}{16} \approx -0.668$

Compute the exact value without a calculator.

(a) $\tan\left(-\frac{3\pi}{16}\right)$

(b) $\cos\frac{3\pi}{16}$

(c) $\csc\frac{2\pi}{9}$

(d) $\sin\frac{\pi}{10}$

(e) $\sec\frac{5\pi}{14}$

(f) $\cot\left(-\frac{3\pi}{22}\right)$

** Exercise 12

Use Proposition 7.1 (ii) and the substitution

$$\theta = 90^\circ - \varphi$$

to prove Proposition 7.1 (i).

** Exercise 13

Use parts (i) and (ii) of Proposition 7.1 to prove parts (iv), (v), and (vi).

** Exercise 14

Suppose $\csc\theta = -17/15$ and $\cos\theta < 0$. Find the six trigonometric functions evaluated at 2θ.

** Exercise 15

Say $\sec\varphi = 5/4$ and $\tan\varphi > 0$. What are the six trigonometric functions evaluated at 2φ?

** Exercise 16

Solve for θ.

(a) $\sin 2\theta + \cos\theta = 0$

(b) $\sin\theta = 1 - \cos 2\theta$

(c) $\cos 2\theta = 3\cos\theta + 4$

(d) $\tan 2\theta + 7 = 7 - \tan\theta$

** Exercise 17

Evaluate without a calculator.

(a) $\cos 165°$

(b) $\tan 285°$

(c) $\sin 22.5°$

(d) $\csc 15°$

(e) $\cot 255°$

(f) $\sec 202.5°$

(g) $\sin 7.5°$

(h) $\cos 191.25°$

** Exercise 18

Compute without a calculator.

(a) $\cos\frac{\pi}{8}$

(b) $\tan\frac{11\pi}{12}$

(c) $\sin\frac{19\pi}{8}$

(d) $\sec\frac{17\pi}{12}$

(e) $\cot\frac{9\pi}{8}$

(f) $\csc\frac{\pi}{12}$

(g) $\tan\frac{\pi}{24}$

(h) $\sin\frac{17\pi}{16}$

** Exercise 19

Say $\tan\theta = -7/24$, $\cos\theta > 0$, and $0 \le \theta < 360°$. Compute the values of the six trigonometric functions at $\theta/2$.

** Exercise 20

Assume $\sec\varphi = -17/8$, $\csc\varphi > 0$, and $0 \le \varphi < 2\pi$. What are the six trigonometric functions at $\varphi/2$?

** Exercise 21

Calculate.

(a) $\cos 105° \cos 45°$

(b) $\sin 30° \sin 15°$

(c) $\sin 105° \cos 105°$

(d) $\cos\frac{435°}{2}\sin\frac{375°}{2}$

** Exercise 22

Evaluate.

(a) $\cos\dfrac{7\pi}{12}\sin\dfrac{\pi}{4}$

(b) $\sin\dfrac{\pi}{6}\cos\dfrac{\pi}{12}$

(c) $\sin\dfrac{29\pi}{24}\sin\dfrac{25\pi}{24}$

(d) $\cos\dfrac{7\pi}{12}\cos\dfrac{\pi}{12}$

** Exercise 23

(i) 1

(ii) $\tan x$

(iii) $-\cot x$

(iv) $\sec x$

Match the above with the expressions below. Some options may be used more than once.

(a) $\dfrac{1+\tan^2 x}{\csc(90^\circ - x)}$

(b) $\cos(90^\circ - x)\csc(x)$

(c) $\cos(-x)\csc(-x)$

(d) $\dfrac{\sin(90^\circ - x)}{\sin(-x)}$

(e) $\cos x \csc(90^\circ - x)$

(f) $\dfrac{\sec x}{\csc x}$

(g) $\dfrac{\sin 2x}{2\cos^2 x}$

(h) $\dfrac{\sin(90^\circ - 2x)}{1 - 2\sin^2 x}$

** Exercise 24

Verify the identity.

(a) $\frac{2\sin(\alpha+\pi/4)}{\sqrt{2}} = \sin\alpha + \cos\alpha$

(b) $\frac{2\cos(\theta-45^\circ)}{\sqrt{2}} = \sin\beta + \cos\beta$

(c) $\tan\left(\gamma + \frac{\pi}{4}\right) = \dfrac{1+\tan\gamma}{1-\tan\gamma}$

** Exercise 25

Verify.

(a) $\dfrac{\cos(90^\circ - \alpha)}{1-\cos^2\alpha} = \csc\alpha$

(b) $\dfrac{\sec(\pi/2 - \beta)}{1+\cot^2\beta} = \sin\beta$

(c) $\dfrac{\cot(90^\circ - \gamma)}{\sec^2\gamma - 1} = \cot\gamma$

** Exercise 26

Verify the identities.

(a) $\dfrac{\tan\alpha}{\tan 2\alpha} = \dfrac{2-\sec^2\alpha}{2}$

(b) $\dfrac{\tan 2\beta}{\sin\beta} = \dfrac{2}{2\cos\beta - \sec\beta}$

** Exercise 27

Verify.

(a) $\sec 2\alpha = \dfrac{\sec^2 \alpha}{1 - \tan^2 \alpha}$

(b) $\csc 2\beta = \dfrac{1}{2} \sec \beta \csc \beta$

(c) $\cot 2\gamma = \dfrac{\cot^2 \gamma - 1}{2 \cot \gamma}$

** Exercise 28

Verify the identities.

(a) $\dfrac{\sin 2\alpha}{1 - \cos^2 \alpha} = 2 \cot \alpha$

(b) $\dfrac{1 - \sin^2 \beta}{\sin 2\beta} = \dfrac{\cot \beta}{2}$

(c) $\dfrac{\sin 2\gamma}{2 - 2\sin^2 \gamma} = \tan \gamma$

** Exercise 29

Verify.

(a) $\dfrac{\cos \alpha + \sin \alpha}{\cos 2\alpha} = \dfrac{\sec \alpha}{1 - \tan \alpha}$

(b) $\dfrac{\cos \beta - \sin \beta}{\cos 2\beta} = \dfrac{\csc \beta}{1 + \cot \beta}$

(c) $\dfrac{\cos 2\gamma + \cos \gamma}{2 \cos \gamma - 1} = \cos \gamma + 1$

** Exercise 30

Verify the identities.

(a) $\dfrac{2 \sin^2(\alpha/2)}{1 - \cos^2 \alpha} = \dfrac{\sec \alpha}{1 + \sec \alpha}$

(b) $\dfrac{\cos^2(\beta/2)}{\sin^2 \beta} = \dfrac{\csc^2 \beta + \csc \beta \cot \beta}{2}$

(c) $\dfrac{\tan(\gamma/2)}{1 - \cos \gamma} = \csc \gamma$

*** Exercise 31

Verify.

$$\sin \theta = \cot \frac{\theta}{2} - \cos \theta \cot \frac{\theta}{2}.$$

*** Exercise 32

Use Theorem 7.1 (i) and (ii) to prove

(a) $\tan(\alpha+\beta) = \dfrac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$

(b) $\tan(\alpha-\beta) = \dfrac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$

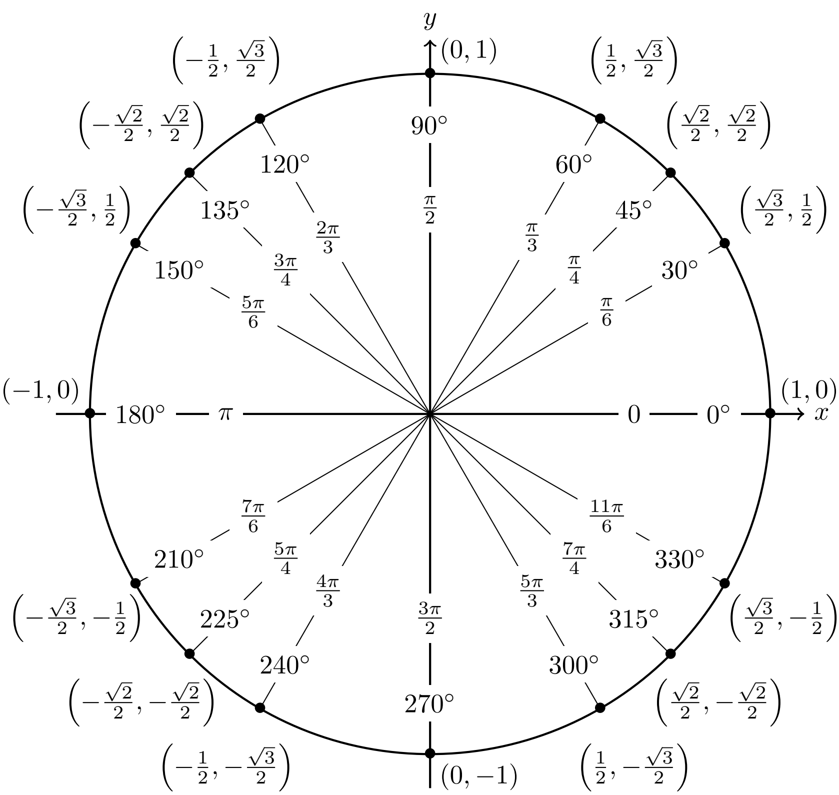
** Exercise 33

Use Theorem 7.1 to prove (a) sine is odd and (b) cosine is even. This exercise is only for didactic purposes; we used that sine and cosine are odd and even, respectively, when we proved Theorem 7.1.

Appendices

Appendix D

Unit Circle



Appendix E

List of Identities

Reciprocal Identities

$$\cos\theta = \frac{1}{\sec\theta}$$

$$\sin\theta = \frac{1}{\csc\theta}$$

$$\tan\theta = \frac{1}{\cot\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Even and Odd Identities

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\sec(-\theta) = \sec\theta$$

$$\csc(-\theta) = -\csc\theta$$

$$\cot(-\theta) = -\cot\theta$$

Pythagorean Identities

$\cos^2\theta + \sin^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Sum and Difference Identities

$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$

$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$

$$\tan(\alpha\pm\beta) = \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$$

Cofunction Identities

$\sin(90^\circ-\theta) = \cos\theta$

$\cos(90^\circ-\theta) = \sin\theta$

$\tan(90^\circ-\theta) = \cot\theta$

$\sec(90^\circ-\theta) = \csc\theta$

$\csc(90^\circ-\theta) = \sec\theta$

$\cot(90^\circ-\theta) = \tan\theta$

Double Angle Identities

$\sin 2\theta = 2\sin\theta\cos\theta$

$$\cos 2\theta = \begin{cases} \cos^2\theta - \sin^2\theta \\ 2\cos^2\theta - 1 \\ 1 - 2\sin^2\theta \end{cases}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Half Angle Identities

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \begin{cases} \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} \\ \dfrac{1-\cos\theta}{\sin\theta} \\ \dfrac{\sin\theta}{1+\cos\theta} \end{cases}$$

Power Reducing Identities

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

Product to Sum and Difference Identities

$$\cos\alpha\cos\beta=\frac{1}{2}\Big(\cos(\alpha+\beta)+\cos(\alpha-\beta)\Big)$$

$$\sin\alpha\sin\beta=\frac{1}{2}\Big(\cos(\alpha-\beta)-\cos(\alpha+\beta)\Big)$$

$$\sin\alpha\cos\beta=\frac{1}{2}\Big(\sin(\alpha+\beta)+\sin(\alpha-\beta)\Big)$$

$$\cos\alpha\sin\beta=\frac{1}{2}\Big(\sin(\alpha+\beta)-\sin(\alpha-\beta)\Big)$$

Inverse Trigonometric Identities

$$\arcsin x+\arccos x=\frac{\pi}{2}$$

$$\arcsin(-x)=-\arcsin x$$

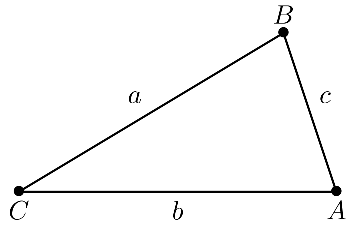
$$\arccos(-x)=\pi-\arccos x$$

$$\arctan(-x)=-\arctan x$$

$$\operatorname{arcsec} x=\arccos\frac{1}{x}$$

$$\operatorname{arccsc} x=\arcsin\frac{1}{x}$$

$$\operatorname{arccot} x=\arctan\frac{1}{x}$$



Law of Cosines

$$a^2=b^2+c^2-2bc\cos A$$

$$b^2=a^2+c^2-2ac\cos B$$

$$c^2=a^2+b^2-2ab\cos C$$

Law of Sines

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

Euler's Formula

$$e^{i\theta}=\cos\theta+i\sin\theta$$

Appendix F

Answers

F.1 Right Triangle Trigonometry

1. (a) 1.072 (b) 0.292 (c) 0.993 (d) 3.716 (e) 0.105 (f) 3.864

2. (a) 0.342 (b) 0.482 (c) 2.351 (d) 14.101 (e) 1.286 (f) 4.900

3. (a) 0.6 (b) 0.75 (c) 0.8 (d) 0.6 (e) 1.333 (f) 0.8

4. (a) 0.555 (b) 0.667 (c) 0.832 (d) 0.555 (e) 1.5 (f) 0.832

5. (a) $b \approx 10.378$
   $c \approx 12.518$
   (b) $a \approx 7.420$
   $c \approx 13.268$
   (c) $a \approx 11.184$
   $b \approx 16.581$
   (d) $b \approx 3.922$
   $c \approx 4.731$

6. (a) $b \approx 21.452$
   $c \approx 28.003$
   (b) $a \approx 4.195$
   $c \approx 6.527$
   (c) $a \approx 65.564$
   $b \approx 78.137$
   (d) $a \approx 1.175$
   $c \approx 1.828$

7. (a) (i) 69.136
   (ii) 44.250

(b) (i) 262.219
(ii) 78.613

(c) (i) 71.407
(ii) 41.716

8. (a) $DF \approx 1.532$
(b) $EF \approx 1.286$
(c) $GH \approx 1.286$
(d) $DG = 2$

9. (a) $DG \approx 14.619$
(b) $DH \approx 13.737$
(c) $EF = 3.75$
(d) $DE \approx 10.964$

10. 309.982

11. 1.407

12. (a) $\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{2}}{2}$ (c) $\frac{\sqrt{3}}{3}$ (d) $\frac{\sqrt{3}}{2}$ (e) $\frac{1}{2}$ (f) $\sqrt{3}$

13. (a) $\frac{1}{2}$ (b) 1 (c) $\frac{\sqrt{3}}{2}$ (d) $\sqrt{3}$ (e) $\frac{\sqrt{2}}{2}$ (f) $\frac{\sqrt{2}}{2}$

14. Answers vary.

15. (a) $UV \approx 6.695$
(b) $VT \approx 8.187$
(c) $TV \approx 15.150$
(d) $ST \approx 11.726$

16. (a) $UV \approx 18.033$
(b) $UV \approx 51.959$
(c) $TV \approx 1.785$
(d) $SV \approx 6.093$

17. (a) $VW \approx 12.483$
(b) $VY \approx 13.486$
(c) The area of $\triangle XYZ$ is about 20.636.
(d) $WZ \approx 12.000$

18. (a) $XZ \approx 21.183$
(b) $VY \approx 26.185$
(c) $XY \approx 57.102$
(d) The area of $\triangle WXZ$ is about 50.150.

19. (a) $\frac{\sqrt{6}-\sqrt{2}}{4}$
(b) $\frac{\sqrt{6}+\sqrt{2}}{4}$
(c) $2-\sqrt{3}$

20. (a) 11.537° (b) 67.792° (c) 78.690° (d) undef (e) 75.522° (f) 18.435°

21. (a) 0.841 (b) undef (c) 0.464 (d) undef (e) 0.795 (f) 1.504

22. (a) 25.377° (b) 78.463° (c) 52.595° (d) 30° (e) 45° (f) 60°

23. (a) 0.287 (b) 0.889 (c) 0.662 (d) 0.785 (e) 0.524 (f) 0.524

24. (a) 24.620° (b) 30.964° (c) 8.213° (d) 56.251°

25. (a) 1.159 (b) 1.134 (c) 0.524 (d) 0.983

26.

(a)

x	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\arcsin x$	30°	45°	60°
$\arccos x$	60°	45°	30°

(b)

x	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\arcsin x$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\arccos x$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$

27. (a) $\alpha = 45^\circ$, $\beta = 60^\circ$, and $\gamma = 30^\circ$.
(b) $\alpha = \pi/4$, $\beta = \pi/3$ and $\gamma = \pi/6$.

28. (a) $m\angle A \approx 65.709^\circ$
(b) $m\angle ACD \approx 15.412^\circ$
(c) $m\angle B \approx 10.963^\circ$
(d) $m\angle A \approx 24.835^\circ$
(e) $m\angle A \approx 53.130^\circ$

29. 15.722

30. (a) 0.176 mi or 931 ft.
(b) 1.015 mi or 5361 ft.

31. (a) 42 cm
(b) 11.254 cm

32. (a) 220.676 ft
(b) 93.262 ft

33. (a) 5.831 ft
(b) 30.968°

34. (a) 6.974 ft
(b) 5.713 ft
(c) 9.567 ft/sec

35. 2.866°

36. 44.427°

37. (a) 347.296 m

(b) 419.550 m

38. 72.471 ft

39. 467.128 m

F.2 Trigonometry of General Angles

1. (a) y-axis
   (b) QIV
   (c) QI
   (d) x-axis
   (e) QIV
   (f) QII
   (g) QII
   (h) QIII

2. (a) QI
   (b) QII
   (c) QIV
   (d) QIII
   (e) QII
   (f) y-axis
   (g) QIV
   (h) x-axis

3. (a) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
   (b) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
   (c) $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
   (d) $(1, 0)$
   (e) $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
   (f) $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
   (g) $(-1, 0)$
   (h) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
   (i) $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
   (j) $(0, -1)$

4. (a) $30°$
   (b) $60°$
   (c) $90°$
   (d) $225°$
   (e) $270°$
   (f) $330°$

5. (a) $\frac{\pi}{4}$
   (b) 0
   (c) π
   (d) $\frac{4\pi}{3}$
   (e) $\frac{5\pi}{6}$
   (f) $\frac{7\pi}{4}$

6. (a) $\frac{\sqrt{2}}{2}$
   (b) $\frac{\sqrt{3}}{3}$
   (c) undef.
   (d) -1
   (e) -1
   (f) $\frac{2\sqrt{3}}{3}$
   (g) 0
   (h) 0

7. (a) $\frac{\sqrt{3}}{2}$
   (b) undef.
   (c) $\frac{1}{2}$
   (d) $\sqrt{2}$
   (e) 0
   (f) $\frac{2\sqrt{3}}{2}$
   (g) $\frac{2\sqrt{3}}{3}$
   (h) undef.

8. (a) $\theta = 90°$
   (b) $\theta = 180°$
   (c) $\theta = 45°$ or $\theta = 225°$
   (d) $\theta = 30°$ or $\theta = 330°$
   (e) $\theta = 60°$ or $\theta = 120°$
   (f) $\theta = 120°$ or $\theta = 300°$

9. (a) $\varphi = \pi/2$ or $\varphi = 3\pi/2$
   (b) $\varphi = \pi/6$ or $\varphi = 5\pi/6$
   (c) $\varphi = 5\pi/6$ or $\varphi = 11\pi/6$
   (d) $\varphi = \pi/3$ or $\varphi = 5\pi/3$
   (e) $\varphi = \pi/4$ or $\varphi = 5\pi/4$
   (f) $\varphi = 7\pi/6$ or $\varphi = 11\pi/6$

10. (a) QI (b) QIV (c) QIII (d) QII (e) QIV (f) QIII

11. (a) Positive x-axis
    (b) Positive y-axis
    (c) Negative x-axis
    (d) Negative y-axis

12. (a) 33° (b) 46° (c) 29° (d) 53° (e) 37° (f) 21° (g) 44° (h) 62°

13. (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{12}$ (c) $\frac{\pi}{10}$ (d) $\frac{\pi}{4}$ (e) $\frac{\pi}{8}$ (f) $\frac{\pi}{6}$ (g) $\frac{\pi}{3}$ (h) $\frac{4\pi}{9}$ (i) $2\pi - 5$ (j) 1

14. (a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) $-\frac{1}{2}$

15. (a) $\frac{\sqrt{2}}{2}$ (b) $-\frac{\sqrt{2}}{2}$ (c) $-\frac{\sqrt{2}}{2}$ (d) $\frac{\sqrt{2}}{2}$

16. (a) $\sqrt{3}$ (b) $-\sqrt{3}$ (c) $\sqrt{3}$ (d) $-\sqrt{3}$

17. (a) $-\frac{\sqrt{3}}{3}$ (b) 0 (c) $-\frac{1}{2}$ (d) undef. (e) $-\sqrt{2}$ (f) $-\frac{\sqrt{3}}{3}$ (g) -2 (h) -1 (i) $-\frac{\sqrt{2}}{2}$ (j) $-\frac{1}{2}$

18. (a) $\dfrac{\sqrt{3}}{2}$
(b) -1
(c) $-\sqrt{2}$
(d) 0
(e) $-\dfrac{2\sqrt{3}}{3}$
(f) $-\dfrac{2\sqrt{3}}{3}$
(g) $-\dfrac{\sqrt{3}}{2}$
(h) $-\dfrac{\sqrt{3}}{3}$
(i) $-\dfrac{\sqrt{2}}{2}$
(j) $-\dfrac{1}{2}$

19. (a) $\sin\alpha = 3/5$, $\cos\alpha = 4/5$, $\tan\alpha = 3/4$, $\csc\alpha = 5/3$, $\sec\alpha = 5/4$, and $\cot\alpha = 4/3$.
(b) $\sin(360^\circ - \alpha) = -3/5$, $\cos(360^\circ - \alpha) = 4/5$, $\tan(360^\circ - \alpha) = -3/4$, $\csc(360^\circ - \alpha) = -5/3$, $\sec(360^\circ - \alpha) = 5/4$, and $\cot(360^\circ - \alpha) = -4/3$.
(c) $\sin(\alpha + 180^\circ) = -3/5$, $\cos(\alpha + 180^\circ) = -4/5$, $\tan(\alpha + 180^\circ) = 3/4$, $\csc(\alpha + 180^\circ) = -5/3$, $\sec(\alpha + 180^\circ) = -5/4$, and $\cot(\alpha + 180^\circ) = 4/3$.
(d) $\sin(180^\circ - \alpha) = 3/5$, $\cos(180^\circ - \alpha) = -4/5$, $\tan(180^\circ - \alpha) = -3/4$, $\csc(180^\circ - \alpha) = 5/3$, $\sec(180^\circ - \alpha) = -5/4$, and $\cot(180^\circ - \alpha) = -4/3$.

20. (a) $\sin\beta = 12/13$, $\cos\beta = 5/13$, $\tan\beta = 12/5$, $\csc\beta = 13/12$, $\sec\beta = 13/5$, and $\cot\beta = 5/12$.
(b) $\sin(\pi - \beta) = 12/13$, $\cos(\pi - \beta) = -5/13$, $\tan(\pi - \beta) = -12/5$, $\csc(\pi - \beta) = 13/12$, $\sec(\pi - \beta) = -13/5$, and $\cot(\pi - \beta) = -5/12$.
(c) $\sin(2\pi - \beta) = -12/13$
$\cos(2\pi - \beta) = 5/13$
$\tan(2\pi - \beta) = -12/5$
$\csc(2\pi - \beta) = -13/12$
$\sec(2\pi - \beta) = 13/5$
$\cot(2\pi - \beta) = -5/12$.
(d) $\sin(\beta + \pi) = -12/13$
$\cos(\beta + \pi) = -5/13$
$\tan(\beta + \pi) = 12/5$
$\csc(\beta + \pi) = -13/12$
$\sec(\beta + \pi) = -13/5$
$\cot(\beta + \pi) = 5/12$.

21. (a) $-\dfrac{15}{17}$
(b) $-\dfrac{8}{15}$
(c) $\dfrac{8}{17}$
(d) $-\dfrac{17}{8}$
(e) $\dfrac{17}{15}$
(f) $-\dfrac{8}{17}$

22. (a) undef.
(b) $-\dfrac{\sqrt{3}}{3}$
(c) $-\sqrt{2}$
(d) -1
(e) 1
(f) -1
(g) $\sqrt{3}$
(h) $\dfrac{2\sqrt{3}}{3}$

23. (a) 1
(b) 0
(c) $\dfrac{2\sqrt{3}}{3}$
(d) -1
(e) $-\dfrac{1}{2}$
(f) $\sqrt{2}$
(g) $-\dfrac{2\sqrt{3}}{3}$
(h) -1

24. Answers vary.

25. (a) D: $\mathbb{R}$ and R: $[-1, 1]$
(b) D: $\mathbb{R}$ and R: $[-1, 1]$
(c) D: $\{x : x \neq \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}, \ldots\}$ and R: $\mathbb{R}$
(d) D: $\{x : x \neq 0, \pi, -\pi, 2\pi, -2\pi, \ldots\}$ and R: $(-\infty, -1] \cup [1, \infty)$
(e) D: $\{x : x \neq \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}, \ldots\}$ and R: $(-\infty, -1] \cup [1, \infty)$.
(f) D: $\{x : x \neq 0, \pi, -\pi, 2\pi, -2\pi, \ldots\}$ and R: $\mathbb{R}$.

26. (a) πn
(b) $2\pi/3 + 2\pi n$ or $4\pi/3 + 2\pi n$
(c) $\pi/3 + \pi n$ or $2\pi/3 + \pi n$
(d) $\pi/2 + 2\pi n$

27. (a) $\pi/18 + 2\pi n/3$ or $5\pi/18 + 2\pi n/3$
(b) $3\pi/8 + \pi n/2$
(c) $1/3 + 2n$, $2/3 + 2n$, $4/3 + 2n$, or $5/3 + 2n$
(d) $2\pi n/5$, $\pi/15 + 2\pi n/5$, or $\pi/3 + 2\pi n/5$

28. (a) 120° or 200°
(b) 15°, 45°, 135°, 165°, 255°, or 285°
(c) 90° or 270°
(d) 105°, 165°, 285°, or 345°

29. (a) $\pi/6$ or $11\pi/6$.
(b) $3\pi/4$
(c) 1, 2, 4, or 5
(d) 1/2, 5/2, or 9/2

30. (a) −135° or −45°.
(b) −90° or 90°
(c) −168°, −24° or 120°
(d) −150°, −90°, −30°, 30°, 90°, or 150°

31. (a) $\cos\alpha = -3/5$, $\tan\alpha = 4/3$, $\sec\alpha = -5/3$, $\csc\alpha = -5/4$, and $\cot\alpha = 3/4$.
(b) $\sin\beta = 8/17$, $\cos\beta = 15/17$, $\tan\beta = 8/15$, $\sec\beta = 17/15$, and $\csc\beta = 17/8$.
(c) $\sin\gamma = -12/13$, $\tan\gamma = -12/5$, $\sec\gamma = 13/5$, $\csc\gamma = -13/12$, and $\cot\gamma = -5/12$.

(d) $\sin\theta = 24/25$, $\cos\theta = -7/25$, $\sec\theta = -25/7$, $\csc\theta = 25/24$, and $\cot\theta = -7/24$.

(e) $\sin\varphi = 0$, $\cos\varphi = -1$, $\tan\varphi = 0$, $\sec\varphi = -1$, and $\cot\varphi$ is undef..

32. (a) $\pi/6 + 2\pi n$, $5\pi/6 + 2\pi n$, or $3\pi/2 + 2\pi n$

(b) $\pi/3 + 2\pi n$ or $5\pi/3 + 2\pi n$

(c) $\pi/4 + 2\pi n$, $3\pi/4 + 2\pi n$, $5\pi/4 + 2\pi n$, or $7\pi/4 + 2\pi n$

33. (a) $2\pi n/3$ or $\pi/3 + 2\pi n/3$

(b) $\pi/6 + \pi n/2$, $\pi/3 + \pi n/2$ or $3\pi/8 + \pi n/2$

(c) $3\pi/20 + \pi n/5$

34. (a) ii, (b) ii, (c) iii, (d) i, and (f) iv.

Answers vary for Exercises 35-41.

F.3 Graphing Trigonometric Functions

1.

	Amplitude	Period	Vertical Shift	Phase Shift
(a)	3	2	0	0
(b)	1/2	360°	1	30°
(c)	2	1	$-\pi$	1
(d)	1	2π	1	π
(e)	2	4π	0	-2π
(f)	3/4	6π	$-7/4$	$-\pi$

2.

(a) $f(x) = -3\sin\left(\frac{\pi}{3}x - \frac{2\pi}{3}\right)$ (b) $g(x) = \pi\cos\left(4x + \frac{2\pi}{3}\right) - 2$

Answers 3-4 omitted to save space.

5. Possible answers:

(a) $A = 1$, $B = 1$, and $C = \pi/2$. (b) $A = 1$, $B = 1$, and $C = -\pi/2$.

6.

	Period	Vertical Shift	Phase Shift	Asymptotes
(a)	$360°$	0	$-90°$	$x = 90°(4n+1)$
(b)	$180/17$	$3\pi/2$	0	$x = 180n/17$
(c)	3π	$2/\pi$	$-\pi$	$x = \pi(6n+1)/2$
(d)	20	1	$-5/6$	$x = 5(12n-1)/6$
(e)	π	$3/4$	π	$x = \pi(2n+3)/2$
(f)	2	1	$3/\pi$	$x = 2n + 3/\pi$

7. Possible answers:

(a) $f(x) = 3\tan\left(\frac{\pi x}{10} - \frac{2\pi}{5}\right) + 1$ (b) $g(x) = -4\cot\left(\frac{x}{5}\right) - 3$

Answers 8-9 omitted to save space.

10. Possible answers:

(a) $A = -1$, $B = 1$, and $C = \pi/2$. (b) $A = -1$, $B = 1$, and $C = \pi/2$.

11.

	Period	Vertical Shift	Phase Shift	Asymptotes
(a)	3	-3	0	$x = 3(2n+1)/4$
(b)	2π	$\pi/6$	$-\pi/3$	$x = \pi(3n-1)/3$
(c)	$\pi/2$	1	π	$x = \pi(n+4)/4$
(d)	4π	0	-2π	$x = \pi(2n-1)$

12. Possible answers:

(a) $f(x) = 3\sec\left(\frac{\pi x}{2}\right) + 2$

(b) $g(x) = -\frac{23\pi}{2}\csc\left(\frac{x}{2} - \frac{\pi}{2}\right) - 11\pi$

Answers 13-16 omitted to save space.

17. Possible answers:

(a) $y = 2\cos\left(\frac{\pi}{2}x + \pi\right) - 1$

(b) $y = -5\sin(\frac{1}{2}x + \pi)$

(c) $y = \cos\left(3x - \frac{\pi}{2}\right) + 2$

(d) $y = \frac{1}{2}\sin\left(\frac{\pi}{4}x + 2\pi\right) - 1$

(e) $y = 2\tan\left(\frac{\pi}{2}x - \pi\right) + 1$

(f) $y = \frac{1}{2}\cot(x + 135°)$

(g) $y = 3\tan\left(\frac{3}{2}x + \frac{\pi}{4}\right) - 2$

(h) $y = \pi\cot\left(\frac{\pi}{8}x - \frac{\pi}{4}\right) + \pi$

(i) $y = 3\sec\left(\frac{\pi}{4}x\right) + 3$

(j) $y = -2\pi\csc\left(2x + \frac{\pi}{2}\right) - \pi$

(k) $y = \frac{\pi}{2}\sec\left(\frac{2\pi}{3}x + \frac{\pi}{3}\right) + \pi$

(l) $y = \frac{1}{2}\csc\left(\frac{\pi}{2}x - \pi\right) - \frac{3}{2}$

Answers 18-19 omitted to save space.

20. (a) 1, (b) 5, (c) 13, (d) 27, and (e) ∞.

F.4 Using Identities

1. (a) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

   (b) $2-\sqrt{3}$

   (c) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$

   (d) $\sqrt{3}+2$

   (e) $-\dfrac{\sqrt{6}+\sqrt{2}}{4}$

   (f) $-\sqrt{6}-\sqrt{2}$

   (g) $-\sqrt{6}-\sqrt{2}$

   (h) $\sqrt{3}+2$

2. (a) $\sqrt{3}-2$

   (b) $-\dfrac{\sqrt{6}+\sqrt{2}}{4}$

   (c) $-\sqrt{3}-2$

   (d) $\dfrac{\sqrt{2}-\sqrt{6}}{4}$

   (e) $\sqrt{6}-\sqrt{2}$

   (f) $\sqrt{2}-\sqrt{6}$

   (g) $\dfrac{\sqrt{6}+\sqrt{2}}{4}$

   (h) $-\sqrt{3}-2$

3. (a) $-\dfrac{\sqrt{3}}{2}$ (d) $-\sqrt{3}$

   (b) -1 (e) $\dfrac{1}{2}$

   (c) $-\dfrac{\sqrt{3}}{2}$ (f) $-\dfrac{\sqrt{3}}{3}$

4. (a) -1 (d) $\dfrac{\sqrt{2}}{2}$

   (b) $-\dfrac{\sqrt{3}}{2}$ (e) 1

   (c) $-\dfrac{\sqrt{3}}{3}$ (f) $-\dfrac{\sqrt{3}}{2}$

5. (a) $\pi/6+\pi n$

   (b) $3\pi/4+2\pi n$ or $5\pi/4+2\pi n$

   (c) $\pi/2+\pi n$

6. (a) $\dfrac{96+5\sqrt{17}}{117}$

   (b) $-\dfrac{40+12\sqrt{17}}{117}$

   (c) $\dfrac{96+5\sqrt{17}}{12\sqrt{17}-40}$

   (d) $\dfrac{117}{96-5\sqrt{17}}$

7. (a) $-\dfrac{84}{85}$ (c) $-\dfrac{85}{36}$

   (b) $\dfrac{13}{84}$ (d) $\dfrac{36}{77}$

8. (a) $y=5\sqrt{2}\sin\left(\pi x+\dfrac{\pi}{4}\right)$

   (b) $y=-6\sin\left(x+\dfrac{\pi}{3}\right)$

   (c) $y=-\sqrt{2}\sin\left(x-\dfrac{\pi}{4}\right)$

(d) $y = 2\sin\left(2x - \frac{\pi}{6}\right)$

9. Answer omitted to save space.

10. (a) 4.331 (b) 0.292 (c) 1.743 (d) 2.145 (e) 0.927 (f) 1.466

11. (a) -0.668 (b) 0.831 (c) 1.556 (d) 0.309 (e) 2.305 (f) -2.190

12. Answers vary.

13. Answers vary.

14. $\sin 2\theta = 240/289$
$\cos 2\theta = -161/289$
$\tan 2\theta = -240/161$
$\sec 2\theta = -289/161$
$\csc 2\theta = 289/240$
$\cot 2\theta = -161/240$

15. $\sin 2\varphi = 24/25$
$\cos 2\varphi = 7/25$
$\tan 2\varphi = 24/7$
$\sec 2\varphi = 25/7$
$\csc 2\varphi = 25/24$
$\cot 2\varphi = 7/24$

16. (a) $\pi/2 + 2\pi n$, $7\pi/6 + 2\pi n$, $3\pi/2 + 2\pi n$, or $11\pi/6 + 2\pi n$

(b) $2\pi n$, $\pi/6 + 2\pi n$, $5\pi/6 + 2\pi n$, or $\pi + 2\pi n$

(c) $\pi + 2\pi n$

(d) $2\pi n$, $\pi/3 + 2\pi n$, $2\pi/3 + 2\pi n$, πn, $4\pi/3 + 2\pi n$, or $5\pi/3 + 2\pi n$

17. (a) $-\frac{\sqrt{2+\sqrt{3}}}{2}$

(b) $-2-\sqrt{3}$

(c) $\frac{\sqrt{2-\sqrt{2}}}{2}$

(d) $\frac{2}{\sqrt{2-\sqrt{3}}}$

(e) $2-\sqrt{3}$

(f) $-\frac{2}{\sqrt{2+\sqrt{2}}}$

(g) $\frac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{2}$

(h) $-\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$

18. (a) $\frac{\sqrt{2+\sqrt{2}}}{2}$

(b) $\sqrt{3}-2$

(c) $\frac{\sqrt{2+\sqrt{2}}}{2}$

(d) $-\frac{2}{\sqrt{2-\sqrt{3}}}$

(e) $1+\sqrt{2}$

(f) $\frac{2}{\sqrt{2-\sqrt{3}}}$

(g) $\frac{2-\sqrt{2+\sqrt{3}}}{\sqrt{2-\sqrt{3}}}$

(h) $-\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$

19. $\sin(\theta/2) = \sqrt{2}/10$
$\cos(\theta/2) = -7\sqrt{2}/10$
$\tan(\theta/2) = -1/7$
$\sec(\theta/2) = -5\sqrt{2}/7$
$\csc(\theta/2) = 5\sqrt{2}$
$\cot(\theta/2) = -7$

20. $\sin(\varphi/2) = 5\sqrt{34}/34$
$\cos(\varphi/2) = 3\sqrt{34}/34$
$\tan(\varphi/2) = 5/3$
$\sec(\varphi/2) = \sqrt{34}/3$
$\csc(\varphi/2) = \sqrt{34}/5$
$\cot(\varphi/2) = 3/5$

21. (a) $\frac{1-\sqrt{3}}{4}$

(b) $\frac{\sqrt{6}-\sqrt{2}}{8}$

(c) $-\frac{1}{4}$

(d) $\frac{\sqrt{2}-1}{4}$

22. (a) $\frac{1-\sqrt{3}}{4}$

(b) $\frac{\sqrt{6}+\sqrt{2}}{8}$

(c) $\frac{\sqrt{3}-\sqrt{2}}{4}$

(d) $-\frac{1}{4}$

23. (a) iv (e) i

(b) i (f) ii

(c) iii (g) ii

(d) iii (h) i

Answers vary for Exercises 24-33.