Problem Set #1

1. A. The key is to notice that the integrand can be rewritten as:
$$\frac{\frac{1}{3}}{\sqrt{1-((\frac{x-3}{3})^2}}$$
Then, we have that the antiderivative of $\arcsin(u)$ where $u = (x-3)/3$.

2. E. Recall $a_0 = f(c)$. Hence, we must find the roots of f. Note that the roots must divide 32 because the polynomial is monic.

3. C. Since X and Y are independent the variance is additive:
$$\sigma_Z^2 = \text{Var }(X+Y) = \text{Var }(X) + \text{Var }(Y) = \sigma_X^2 + \sigma_Y^2$$

4. B. The number of bijective functions is $4!$. Using the inclusion-exclusion principle, we have that:
$$4! - \left(\begin{array}{c}4\\1\end{array}\right)\cdot 3! + \left(\begin{array}{c}4\\2\end{array}\right)2! - \left(\begin{array}{c}4\\3\end{array}\right)\cdot 1! + 1 = 9$$

5. B. The integrand is $e^x + e^{-x}$

6. B. (a) is hyperbolloid of one sheets, (b) is a hyperbolloid of two sheets, (c) is a sphere, (d) is a saddle, (e) is a plane. What you should look for is a plane that divides your surface into two distinct parts. in this case, it's the yz-plane (i.e when x = 0). Note there are no solutions there!

7. E. I can be shown to be true by looking at the function along $y = x^2$. Along this curve, $f = 1/2$. III is true away from the origin and is true at the origin to since $f = 0$ along the x- and y-axes.

8. E. $5^2 = (5^n)^{2/n} \le (3^n + 5^n)^{2/n} \le (2 \cdot 5^n)^{2/n}$. Then apply the sandwich theorem.

9. C. I is false when $T = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}$. II is false because the zero matrix is diagonalizable so T^{-1} doesn't exist. III is true by the spectral theorem. IV is false as $\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$ (this matrix is in Jordan form; but you could also check that it's not diagonalizable by hand).

10. C. The first trick is to rewrite ths integral as:
$$I_1 = \int_0^{\pi/2} \frac{\cos x}{\sin x + \sin x} dx$$
Next consider
$$I_2 = \int_0^{\pi/2} \frac{\sin x}{\sin x + \sin x} dx$$
Add and subtract these two integrals from each other to get integrals you can actually do. For instance,
$$I_2 - I_1 = \int_0^{\pi/2} \frac{\sin x - \cos x}{\sin x + \sin x} dx = \ln(\sin x + \cos x)|_{x=0}^{\pi/2} = 0$$
Easily, $I_1 + I_2 = \pi/2$. From here we see $2I_2 = \pi/2$ and then get our answer.

Problem Set #2

1. C. The series is the $\arctan(1)$. Might be a good idea to be able to derive this from the geometric series without worrying about convergence.

2. D. Restricting f to $[0,1]$, we see that f must attain its min and max. We want to apply the theorem that a continuous function on a compact domain is *uniformly* continuous. We can't just use $[0,1]$ because we don't know how $.99$ and $.01$ compare. So, we just enlarge our space of consideration, namely to $[0,2]$ and apply the theorem.

   To get III and IV, use the auxiliary function $g(x) = f(x) - f(x+\pi)$. This is continuous and since f attains maximum and minimum values, call them x and y, respectively. then $g(x) \geq 0$ and $g(y) \leq 0$. To finish, apply the IVT.

3. B. You could let $|S| = 3$ and count the number of non-surjective functions, which is 21. B is the only answer choice that works. Alternatively, you could note that what we want to do is first choose 2 elements of the 3 that will be "hit" and then count the number of functions into a two element. However, we have overcounted by 3 the number of functions in which all elements are sent to a single element in $\{1,2,3\}$.

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} 2^{|S|} - 3$$

   Alternatively, use the inclusion-exclusion principle:

$$2^{|S|} - (2^{|S|} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} 2^{|S|-1} + 3)$$

4. B. It's a good idea to be able to go through the definitions and prove I and II. Also, III is false, but true when X, Y are independent because if $\max(X,Y) < 2$, then both $X < 2$ AND $Y < 2$. Then apply the definition of independence. However, if $X = Y$, then $\max(X,Y) = X$ and the statement is false. IV is false because if $Y = -X$ where X is your favorite Random Variable (work this out!).

5. B. The image of a compact set must be compact, so I can't happen. II can happen (draw a picture to convince yourself). III can't happen because f_3 attains it's maximum value on the interior, then it cannot be injective!

6. C. It amounts to finding the angle between the two normals of the planes, namely (2, 2, 4) and (0, 0, 1). This is done by the formula $\theta = \arccos\left(\frac{(2,2,4)\cdot(0,0,1)}{||(2,2,4)||\cdot||(0,0,1)||}\right)$

7. E. We have to verify the following properties:

   Proposition. *Let S be a commutative ring, and let R be a subset of S. Then R is a subring of S if and only if:*

   (i) R is closed under addition and multiplication

   (ii) if $a \in R$, then $-a \in R$ (this implies that we have the additive identity in S)

(iii) R contains the multiplicative identity of S.

Note that I fails (i) because $x^2 + x$ and x^2 are both even degree but their difference is odd degree. II fails (ii) because x^3x^3 has degree bigger than 3. III fails (iii) because the identity has an odd coefficients. III is an ideal, though, meaning it satisfies the first two properties, but not the last.

8. A. It's long. The integral is $\int_1^2 2\pi e^x\sqrt{1+e^{2x}}dx$. It requires first a u-sub, then a trig. sub, and then to compute $\int \sec^3$ you need to perform integration by parts and know $\int \sec$. It's involved.

9. B. The left is linear and the determinant in terms of k should be $(k+2)(k-1)^2$. For $k=1$, there are infinitely many solutions, while for $k=2$ there none since adding the last two equations yields $2x - y - z = -2$, which is inconsistent with the first. Lots of computations here!

10. Similar to the previous #10. Rewrite the integral as:

$$I_1 = \int_0^{\pi/2} \frac{\cos^e x}{\sin^e x + \sin^e x} dx$$

Use the substitution $u = \pi/2 - x$. Then note this integral becomes:

$$I_2 = \int_0^{\pi/2} \frac{\sin^e x}{\sin^e x + \sin^e x} dx$$

Hence, $I_1 = I_2$. So, we find $I_1 + I_2 = \pi/4 = 2I_1$. We could have replaced the e with any real number!!!

Problem Set #3

1. D. III is closed and bounded. Bounded is easy because each of the columns have norm less than 1 (might want to expand what it means to be bounded and make sure you can prove this). Closed is straight forward because if A is an orthonormal matrix with columns $\vec{x}, \vec{y}$, then consider the continuous function $g(A) = \langle \vec{x}, \vec{y} \rangle$. Using properties of continuity, we see that if $\{A_n\}$ is some sequence of orthogonal matrices with limit B, then B must have orthogonal columns. Similar, arguments work for the norm of each column.

   To see that I and II are not compact use:

$$A_n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

2. C. Need that $\ln n < n^l$ for any $l > 0$ for sufficiently large n; that is there is some N depending on l for which $n > N$ implies the inequality holds (prove this using L'Hospital's rule if you like). Then go far enough out in the series (details!).

3. D. Some relevant facts:

   - The finite cartesian product of countable sets is countable.

- Similarly, a finite cartesian product set of infinite sets each of the same cardinality has the same cardinality.
- The countable union of countable sets is countable.
- The countable cartesian product of countable is uncountable (Canter's diagonal argument)

All the sets are uncountable except for (d), which is the countable union of countable sets.

4. B. Use the Limit Comparison Test to see $\sin(\pi/n) \sim 1/n$.
5. B. Make the sub $u = \sqrt[3]{x}$, then do integration by parts. Or just integrate the answers!!!
6. A. Differentiate $f(t) = \langle \vec{r}(t), \vec{r}(t) \rangle$ and note it is bigger than 0.
7. B. Check the derivative and note its an increasing function so bijective.
8. B. L'Hospital's rule and the fundamental theorem of calculus.
9. D. Stars and Bars: $\begin{pmatrix} 7 \\ 3 \end{pmatrix} = 35$.
10. D. Right as a fraction and then note that $L = \lim_{n \to \infty} x_{n-1}/y_{n-1}$ and so divide numerator and denominator by y_{n-1} and use the standard trick.

Problem Set #4

1. A. A subgroup of index 2 is always normal. Let H be a subgroup of index 2. The left- and right-multiples of H partition G. A set of index two means there are only two subgroups in each partition: $\{H, aH\}$ and $\{H, Ha\}$. Hence, $Ha = aH$ and H. Hence I cannot happen. II happens when G is abelian. III happens when $G = \mathbb{Z}/7\mathbb{Z}$, say.

2. E. Lots of different ways. One way is to use the chain rule:

$$\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{dy}{dx}\frac{d^2x}{dt^2}$$

and we know $\frac{dy}{dx} = (\frac{dy}{dt})/\frac{dy}{dt}$. So:

$$\implies \frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2} - \frac{dy}{dx}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2}$$

and

$$\frac{-\sin t - \cos t}{e^{2t}}$$

3. Write everything over a common denominator say n and note any subgroup H looks like $H = \frac{1}{n} \cdot \widetilde{H}$, where $\widetilde{H}$ is a subgroup of $\mathbb{Z}$. Since $\mathbb{Z}$ is a PID (aka every subgroup is cyclic), we have $\frac{m}{n}\mathbb{Z} = \widetilde{H}$. II is true as $G = \{m/2^n : m, n \in \mathbb{Z}\}$. III is true since every element has infinite order.

4. E. Note the equilibrium points are $y \equiv \pm 1$, so by uniqueness theorems must stay in between them. Second. Note that the derivative of the left side is negative, so -1 is a stable equilibrium point.

5. B. Partial fractions produce a telescoping sum.

6. D. I is false because you need the first derivatives to be equal, too.

7. D$y' = .4 \cdot 2 - \frac{y}{50} \cdot 2 \implies S' + y/25 = .8$. Hence the solution can be solved by an integration factor $e^{t/25}$ and you get $20 + Ce^{-t/25}$. Since $S(0) = 20$, we have:

$$y = 20 - 20e^{-t/25}$$

Setting $y = 10$ and solving gives the answer.

8. B. Pretty hard, but if you have taken complex analysis, you might think this looks a lot like $\int_{|z|=1} \frac{e^z}{z} dz$ and hence by the Cauchy Integral Formula is $2\pi i e^0$. The key is to draw the picture in the complex plane and then figure out a complex integral for it. The parameterization $\phi(\theta) = e^{i\theta}$, so $dz = ie^{i\theta} d\theta$ and:

$$\int_{|z|=1} \frac{e^z}{z}\, dz = \int_0^{2\pi} \frac{e^{e^{i\theta}}}{e^{i\theta}} \left(ie^{i\theta}\, d\theta \right)$$

9. D. Make the substitution $y/\sqrt{t} = x$.

10. D. One can use taylor expansion of $f(x+t)$ around x or simply apply $f(u) = x$, then:

$$\begin{aligned} \lim_{\varepsilon \to 0} \frac{3}{\varepsilon^3} \int_{-\varepsilon}^{\varepsilon} t(x+t)\, dt &= \lim_{\varepsilon \to 0} \frac{3}{\varepsilon^3} \int_{-\varepsilon}^{\varepsilon} xt^2 + t\, dt \\ &= \lim_{\varepsilon \to 0} \frac{3}{\varepsilon^3} \left[\frac{xt^3}{3} + \frac{t^2}{2} \Big|_{-\varepsilon}^{\varepsilon} \right] \\ &= 2x \end{aligned}$$

Well, $f'(x) = x$ and so the answer should be $2f'(x)$.

Problem Set #5

1. B. Have to maximize the function $f(\theta) = \cos^2\theta + 2\sin\theta\cos\theta + 2\sin^2\theta = 1 + \sin 2\theta + \sin^2\theta$ and so setting the first derivative to be zero: $0 = f'(\theta) = 2\cos 2\theta + 2\sin\theta\cos\theta = 2\cos 2\theta + \sin 2\theta$. That is $\tan 2\theta = 2$.

2. C. Use $f(x) = (\ln x)/x$ and maximize.

3. C. Set $G(a,b,c) = \int_b^a e^{t^2+ct} dt$. Then $F(x) = G(\cos x, \sin x, x)$ and apply the chain rule to get:

$$F'(x) = e^{\cos^2 x + x\cos x}(-\sin x) - e^{\sin^2 x + x \sin x}(\cos x) + \int_{\sin x}^{\cos x} te^{t^2+xt} dt$$

and

$$F'(0) = -1 + \int_0^1 te^{t^2} = -1 + \frac{e}{2} - \frac{1}{2} = \frac{1}{2}(e-3)$$

4. B. II requires some branch of the ln and III only converges on the interior of a unit disk.

5. E. For I, an entire function with positive real part mean that e^{-f} is bounded. For II, by adjusting f so that L is imaginary axis, i.e. $\alpha f + \beta$, we can assume the image of f has positive real part or negative real part as in I.

   Could also use Picard's theorem, which says that entire functions can only miss at most a single point in their image (for instance e^z misses 0).

   For III, by the maximum modulus principle we know that $|f(z)/z|$ takes its maximum on the boundary of disk. Couple this that its getting closer to zero shows $f(z)/z = 0$. Hence, $f(z) = 0$.

   IV is easily nonconstant because we don't assume the function is entire. If it was, apply Picard's theorem.

6. D. He must make the last free throw!

7. D. I is false; take $R = \mathbb{Z}$. II is true as $(ab)^2 = ab = a^2b^2$. Yes, since $a^2 = a$ implies $a = 1$ by cancellation.

8. B. It clearly has an $n-1$ dimensional kernel and an eigenvalue n. One could inspect the case $n = 2$ and compute the characteristic polynomial explicitly.

9. D. Divide top and bottom by $1/n$ and notice it is a Riemann sum.

10. C. The key to this problem is picking the right "reference frame". Let's look at the frame of the hour hand, i.e. we are sitting there and watch the minute hand go around. Every revolution in this reference frame corresponds to two right angles. In a 12-hour period, the clock makes one revolution and the minute hand makes 12, but end up at the same original positions. That means 11 revolutions with respect to our special reference frame were made. Thus, 22 right angles in a 12-hour period. In a 24 hour period, 44 angles are obtained.

    Even more formally, let $\theta(t) :=$ the <u>total</u> radians the minute hand travels in the reference frame t hours after 12 o'clock. Specifically, $\theta(0) = 0$. To count the revolutions in this reference frame is to calculate $\theta(24)/2\pi$. We note that $d\theta/dt = m$, where m is constant (this follows since both minute and hour hand move at constant rates). So, $\theta(t) = mt$. Now, m must be calculated. Well, $\theta(1) = (11/12) \cdot 2\pi = 11\pi/6 = m$ (this is one o'clock). So,

$$\begin{aligned} \theta(24) &= mt = \frac{11\pi}{6} \cdot 24 = 44\pi \\ \frac{\theta(24)}{2\pi} &= 22 \end{aligned}$$

    The number of right angles is thus 44, which we came up with.

Problem Set #6

1. C. Let (b, c) be the point where the parabola hits the line a second time. The shaded area is then $\int_0^b ((2x - 3x^3) - c)dx = 0$. Also, use that $c = 2b - 3b^3$. Then do some algebra.

2. C. I is true, since $\text{curl}\ \text{grad}\ f = 0$. II s false since $\text{div}\ \text{curl}\ \vec{F} \neq 0$. III is true just take the time derivative and check.

3. I is true as similar matrices have the same eigenvalues and $A^{-1}ABA = BA$. II is false. I don't have a good example except to say that you should go on wolfram alpha and convince yourself this is not the case (plug in two symmetric matrices). So, even though AB and BA will have the same eigenvalues, there eigenvectors will be distinct in general. III is false (use jordan blocks of varying size corresponding to eigenvalue 3).

4. D. Use spherical coordinates and that z must be positive to deduce that $r^4 = 2r^3 \cos\phi \sin^2\phi$ and:

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\phi\sin^2\phi} r^2 \sin\phi d\theta = \frac{16\pi}{3}\int_0^{\pi/2} \underbrace{\sin^7\phi\cos^3\phi}_{(\sin^7\phi - \sin^9\phi)\cos\phi}\ d\phi$$
$$= \frac{2\pi}{15}$$

5. B. Conditional probability

6. D. The characteristic polynomial has no repeated roots, and so the matrix is diagonalizable. Since $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \ldots + 1)$, we have that if λ is a not a real eigenvalue, then the right factor vanishes. For III, just take a 2×2 diagonal matrix with i along diagonal.

7. C. The theorem: If D is simply connected (in $\mathbb{R}^2$ or $\mathbb{R}^3$) and $\vec{X}$ a vector field on D, then $\vec{X}$ is conservative if and only if $\text{curl}\ X = 0$ if and only if $\vec{X} = \text{grad}\ f$ for some function $f : D \to \mathbb{R}$.

8. D. Let s be the side length of the cube: $\frac{s\sqrt{2}}{2} = \frac{3-s}{3}$.

9. E. Note we have:

$$\int_0^\infty \int_a^b e^{-xy} dydx$$

   Then apply Fubbini.

10. C. There are two possible empty seats when the last person arrives: the drunkard's or the last person to arrive. The number of arrangements in each scenario is equal (a explicit bijection is formed by switching people between the two aforementioned seats). Thus, the probability is 1/2.

Problem Set #7

1. B. Check the curl vanishes.

2. D. Stars and bars.

3. E. It's easiest to find a matrix that is not diagonalizable by looking at Jordan normal form. The other matrices are easily seen to be. Note the eigenvalues occur along the diagonal and if a matrix has two distinct eigenvalues, then it is diagonalizable.

4. E. $u_x = f'(x-ut)(1-tu_x) \implies u_x = f'/(1+t)$, $u_t = -f'(x-ut)\cdot(u+tu_t) \implies u_t = f'u/(1+t)$
5. E. Lagrange multipliers.
6. C. Related Rates
7. B. $\frac{1}{2}b(1-h/2) = bh$
8. A. The only possible digits are $2, 4, 8, 6$ and cycle in that order upon each multiplication of 12. In fact, $12^{4n} \equiv 2$ for any n.
9. E. Look below line $y = x$ and get $\int_0^1 \int_0^x e^{x^2} dydx$, then multiply by 2.
10. E. If $y > 0$, the answer is A (try it). This is a convolution of two exponential random variables with $\lambda = 2$.

Problem Set # 8

1. D. The normals of the planes are orthogonal
2. D. Substitute $x^2 = 3 - x$ into the equation twice.
3. E

$$\begin{aligned}\frac{d}{dx}\ln(\ln x^{x^2+x}) &= \frac{d}{dx}\ln((x^2+x)\cdot\ln x)\\ &= \frac{(2x+1)\ln x + (x+1)}{(x^2+x)\ln x}\end{aligned}$$

Setting $x = e$ we get:

$$\frac{3e+2}{e^2+e}$$

4. A. It's an increasing function.
5. D. Replace y with $\sqrt{y^2+x^2}$. Alternatively (thanks, Ben!), find points like $(0, 2, 0)$ and $(2, 0, 0)$ on the torus and plug in. Note (a) can't be right either because it contains the origin.
6. E. $96 = 3 \cdot 2^5$
7. D. Surefire way is to let $P(x) = x^2$ and then substitute. Alternatively, we can write the quotient as:

$$\lim_{h\to 0}\frac{\frac{P(x+h)-P(x)}{h} - \frac{P(x)-P(x-h)}{h}}{h}$$

8. D. The integral is $= 2\pi i$(residue); where the residue is the coefficient of $1/z$. Note the first function is holomorphic and the second function we substitute $1/z$ into the power series for the cosine function.

9. D. This is a change of variables:
$$\int_U f(\phi(\vec{x})) = \int_{\phi^{-1}(U)} f(\vec{y}) \det(D\phi^{-1})$$
where $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism. $\phi(x,y) = (x+y, x-y) = (u,v)$ and then our integrand becomes ue^{uv}. We see that $\det(D\phi) = 2$, so $\det(D(\phi^{-1})) = \det((D\phi)^{-1}) = \frac{1}{2}$ and the integral becomes:
$$\frac{1}{2}\int_0^3 \int_0^2 ue^{uv} dv du = \frac{1}{2}\int_0^3 (e^{2u} - 1) = \frac{1}{2}(\frac{e^6}{2} - \frac{1}{2} - 3)$$
10. D. $101110 - 11 = 101011$ and this is 43. We are missing an endpoint, so this means the number of digits is 44.

Problem Set #9

1. C. $\nabla f = (2x + 3y, 3x + 2y + 4y^3)$ and clearly $(0,0)$ is a critical point. It has another pair of critical points setting $x = -\frac{3x}{2}$ and getting $y = \pm\frac{\sqrt{10}}{2}$. Now,
$$H_f = \begin{pmatrix} 2 & 3 \\ 3 & 2 + 12y^2 \end{pmatrix}$$
At $y = x = 0$, $|H_f| = -5 < 0$ and hence is a saddle (negative eigenvalue and positive eigenvalue).
2. E. Could use Lagrange interpolation, but probably easier just to plug in.
3. C. Use stokes and integrand becomes 3 and then just need to find area of the triangle.
4. B. II is false by inspecting $f_n = g_n = x + \frac{1}{n}$. III is true since the sine function is uniformly continuous. IV is clearly
5. C $(27)13 + (-10)35$
6. A. So this mean $\nabla w \cdot (2, -3) = 0$. But $\nabla w \perp$ level curves of w, that is when $w(x,y) = c$. Hence $(2,-3)$ is parallel to curves $w(x,y) = c$ for all such x, y! This means that $2x + 3y$ =constant should BE level curves of w. Clearly, D is the only thing that works.
7. D. $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $(\mathbb{Z}_2)^4$
8. B. Compare with $1/n^2$
9. C. Use functions that have thinner and thinner width with the increasing height as $x \to \infty$. Heuristically, the function should have humps at each $n \in \mathbb{N}$ with width $(1/4)^n$ at each height 2^n. Smooth this out.
10. A.
$$\arctan\left(\frac{n+1-(n)}{1+(n)(n+1)}\right) = \arctan(\tan(\theta_{n+1} - \theta_n) = \theta_{n+1} - \theta_n = \arctan(n+1) - \arctan(n)$$
So, partial sums are $\arctan(n) - \pi/4$ and the result follows since the series is telescoping!

Problem Set #10

1. C. Morera's theorem implies the "I" if false because if holomorphic functions converge to another continuous function under this metric (i.e. converge uniformly), then the continuous function must have integral zero (uniform convergence implies limits and integration commute). Clearly, there are continuous functions that have non-zero path integral. II is false ($f_n(x) = x^n$). It is false even when it converges to a a continuous function! (take the disappearing hump). III is true simply by writing out the fundamental theorem of caluculus. The complete details can be found for instance in Strichartz, Way of Analysis pg. 262 or Tao, Analysis II pg. 464

2. D. I is false whe $A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. II is true by multiplicativity of the determinant. III is false when $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

3. C. The relevant theorem is:

   Theorem. *If $y' = f(t, y)$ is differential equation with initial condition $y(t_0) = y_0$, then there exists a solution a solution if f is continuous in the ty-plane in a neighborhood of (t_0, y_0). Uniqueness is obtained if f is lipschitz in the y variable in a neighborhood of (t_0, y_0) (easier to just check if $\frac{\partial f}{\partial y}$ is continuous).*

   Let's suppose $a = 0$. Note that $y = 0$ and $y = \frac{t^2}{4}$ is a solution. We can actually amalgmate these solutions. For $C \geq 0$ we have solutions:

   $$y_C(t) = \begin{cases} 0 & x \leq C \\ \frac{1}{4}(x - C)^2 & x \geq C \end{cases}$$

   However, for $a \neq 0$, the theorem above holds.

4. E. Note the power series is $\frac{1}{2}\cos 2 = \sin 1 \cos 1$.

5. C. Use polar coordinates. The first circle is $r = 2\cos\theta$ and the second is $r = 2\sin\theta$. The area is then easiest split in half from $\theta = \pi/4$ to $\pi/2$ and from $\pi/4$ to $\pi/2$. These have the same area by symmetry:

   $$2\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta}(rdrd\theta) = -\int_{\pi/4}^{\pi/2} 4\cos^2\theta d\theta = (2 + \sin 2x)|_{x=\pi/4}^{\pi/2} = \pi/2 - 1$$

   Yet another way (thanks Devin), is to use basic geometry and symmetry. Essentially cut out a quarter circle with the square. Key point is the circles intersect at (1, 1)

6. B. View this part of the sphere as rotation about the top of the y-axis from the viewpoint of the xy-plane. So, $2\pi\int_h^R \underbrace{\sqrt{R^2 - y^2}}_{y} \cdot \underbrace{\sqrt{1 + \frac{y^2}{R^2 - y^2}}dy}_{ds} = 2\pi R(R - h)$

7. D. We are finding the area in the square $[0,1]\times[0,1]$ underneath the parabola $y=x^2$ which is precisely $1-1/3$.

8. E. Take the ln and then apply L'Hospital's rule a bunch. An easier approach (thanks John), is to taylor expand $\sin x/x$ and then do the same trick only things work out much simpler.

9. E. Use the root test on:
$$\frac{n!}{2^{n^2}} \leq \frac{n^n}{(2^n)^n} = \left(\frac{n}{2^n}\right)^n$$

10. B. $\lfloor 100/\pi \rfloor = 31$. Let $\pi = 22/7$. Can you do this with cosine?

Linear Algebra #1

1. A. Multiply the left matrix by the right matrix to simplify calculations: $6+c-3=0$

2. B. Get the first two rows of the matrix in the form:
$$\begin{pmatrix} 2 & 3 & 4\dots \\ 1 & 1 & 1\dots \end{pmatrix}$$
and these two rows span all the others.

3. D. It has eigenvalue -1, but no zero eigenvalue. Hence, II is false, but the others are true.

4. E. The diagonal must be zero, and the entries above the diagonal determine those below. Count these: $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$.

5. A. Take the determinant OR (thanks Devin) just notice that $7-6+1=0$, $5-4=x$, $8-7+1=0$ $1=1$. That is the first column minus the second column plus the fourth column is the third.

6. C. The matrix is diagonalizable over $\mathbb{C}$ because it has two distinct eigenvalues: $\pm i$.

7. D. The characteristic polynomial is x^2-6x+9. Since the eigenspace is 1 dimensional, we see the answer must be D.

8. E. I is true by Cayley Hamilton (note: the roots of the minimal polynomial and the characteristic polynomial are the same only with different multiplicity). II is by the spectral theorem and I. III is false (use the zero matrix). IV is true as if $ST^n=0$, then TS^{n+1} is too.

9. B. $V=\mathrm{span}(1,x,x^2,x^3)$ and $W=\mathrm{span}(x(x-1)(x+1))$.

10. E. Product rule!

Linear Algebra #2

1. E. Use $\det(AB) = \det(A)\det(B)$ to see I. To see II, note that 0 is not a root of the characteristic polynomial. III is true by direct computation of the operator $3A^2 - \lambda A$.

2. C. Should get $t = 0, 1$. Get two choices for A and then just inspect.

3. C. Remember we view $n \times n$ matrices as $\mathbb{R}^{n^2}$. We use that all complex matrices have a basis in which they are upper triangular. Perturb the diagonal entries so that they are either nonzero (I) or all distinct (II). The others are not true. For II, if $\det A \neq 1$, then there is an open set $U \subset \mathbb{R}$ such taht $\det A \in U$ and $1 \notin U$. Hence, $\det^{-1}(U)$ is an open set which excludes unitary matrices around A. Symmetric matrices are clearly not dense.

4. E. Note by inspection $(1, \ldots, 1)$ is an eigenvector for $nb - b + 1$. Similarly, $(1, -1, 0, \ldots 0)$ and $(1, 0, -1, 0, \ldots, 0)$, etc. are eigenvectors for $b - 1$.

5. B. Need to know what rotation matrices look like in 3-space.

6. C. We already know the rank is 2 (see Linear Algebra # 1 problem 2).

7. C. If $f(x) = \sqrt[3]{x}^3$, then f is not additive. If $f = 0$, then $f(1) = 0$, too. C is true because f is linear and hence must be surjective. To see this, $f(m+n) = (m+n)f(1) = mf(1)+nf(1) = f(m+n)$ when $m, n \in \mathbb{Z}$. Then for rational points, assume a common denominator: $f(m/n + p/n) = \frac{1}{n} \cdot f(m+p) = \frac{1}{n}f(m) + \frac{1}{n}f(p) = f(m/n) + f(p/n)$. Since f is continuous, this extends to all real numbers by taking limits.

8. B. Use:

   Proposition. *Let V be a vector space over a field F. A subset $U \subset V$ is a subspace if the following conditions are satisfied:*

   (a) (Additive Identity) $\vec{0} \in U$

   (b) (Closure under Addition) $\vec{u}, \vec{v} \in U$ implies $\vec{u} + \vec{v} \in U$

   (c) (Closed under scalar multiplication) $a \in F$ and $\vec{u} \in U$ implies $a\vec{u} \in U$.

Abstract Algebra

1. A. All you need to know is that the order of a subgroup divides the larger group. In this case if we call L the subgroup generated by H and K, then $|L| \leq 360$. Note that H and K must be contained in L, so 12 must divide $|L|$ and doesn't divide A.

2. D. The two numbers are not relatively prime.

3. D. Need two know $(ab)^2 = a^2b^2 \implies ab = ba$. This is similar to the proof that also comes up a lot: If G is a group such that every element has order 2, then the group is commutative. To prove this: $(ab)^2 = ab = a^2b^2$.

4. E. Need $f(e) = e$, so $a^3 = e$.

5. A. Use $\mathbb{Z}/9\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ for examples. A is false because it means $\langle x \rangle$ has size 2, which is impossible since 2 does not divide 9.

6. E. You chould check or note that I and II are fields (really just adjoining roots–this is basic field theory). III is a subring by inspection. IV is not closed under addition.

7. C. Could actually use the answers. Know the group is divisible by 7 and cannot be even because counting shows that groups of even order have elements that are their own inverse.

8. C. For p^2 there are two: $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p \times \mathbb{Z}_p$. For q^4, there are 5: $\mathbb{Z}_{q^4}, \mathbb{Z}_{q^3} \times \mathbb{Z}_q, \mathbb{Z}_{q^2} \times \mathbb{Z}_{q^2}, \mathbb{Z}_{q^2} \times \mathbb{Z}_q \times \mathbb{Z}_q, \mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_q$. Then the answer is multiplying them.

9. D. Use the quarternions to see I is false (it's like a the vector space $\mathbb{R}^3$ except multiplication is defined using the cross product; look it up though). Use $\mathbb{Z}/7\mathbb{Z}$ to see why III is false.

10. E. Note the ring must have characteristic 2 because if we look at the homomorphism $\varphi : \mathbb{Z} \hookrightarrow S$ that sends $1_\mathbb{Z}$ to 1_S, we know that $(1_S + 1_S) = (1_S + 1_S)^2$. That means that $2 \in \ker \varphi$. This easily tells us I. II and III are intimately related because:

$$s^2 + ts + st + t^2 = (s + t)^2 = s + t = s^2 + t^2$$

hence, $st = -ts$, but we have characteristic 2. Hence, II and III are true.

Number Theory

1. C. Need to count relatively prime integers $\phi(24) = \phi(3)\phi(2^3) = (3 - 1)(2^2)(2 - 1) = 8$, where ϕ is the Euler-totient function.

2. E. Without doing any serious computations recall the important fact that if $(a, b) = d$, then there are integers $m, n \in \mathbb{Z}$ such that $ma + nb = d$. That instantly excludes a, b, c, d. Also e is the cyclic grou generated by p.

3. C. 1 must go to 0 or an element of order 3.

4. D. x = 5, y =1 works and then plug in.

5. D. $7^{25} \equiv 7^1 \equiv 7 \mod 10$ because $\phi(10) = (5 - 1)(2 - 1) = 4$.

6. C. Count the number of prime factors. P_1 will always have exactly one while P_{23} will always have two, etc.

7. D. It's a debate between 0 and 5. 100! has $10 * 2 + 1 + 1 = 22$. So, $400!$ has 88. Need to add 11 more zeros to this. However at 450! we get 2*5+1

Real Analysis and Advanced Calculus

1. D. To see II, if II is not compact consider $f(x) = 1/|x-y|$, where y is a limit point not in K. III is clearly false (two disjoint boxes) and I is true because it's one of the most fundamental theorems in real analysis!

2. D. We see that $f_n \to 0$ on $[0,1)$ and $1/2$ for $x = 1$. Therefore II must be false. III is true inspite of II because:
$$\frac{1}{2(n+1)} \leq \int_0^1 f_n(x)dx \leq \frac{1}{n+1}$$

3. E. Just note that $n!/n^n \leq 2/n^2$ and then apply the ratio test.

4. E. Use the intermediate value theorem $x = 0$ and $x = 1$. II doesn't work because $x^2 + 2$. Note it's on (0,1)!

5. D. Have to draw the picture!

6. C. Let $F(x) = g(x) - f(x)$ and therefore F is non-decreasing.

Topology

1. B. Has to be closed under unions and finite intersections (look up the definition of a Topological space). The power set and $\{\{a,b\},\emptyset\}$ are two. $\{\{a,b\},\emptyset,\{a\}\}$, $\{\{a,b\},\emptyset,\{b\}\}$ are two more.

2. D. Intersections are singeltons and hence does not satisfy the conditions for a topology.

3. E. $\omega(0,1/2) + \omega(1/2,1) = 1/2 < 1 = \omega(0,1)$. Good exercise to prove the other are especially III. The way to do this is to set $a = d(x,y)$, $b = d(y,z)$ and $c = d(x,z)$ and use that we know $a + b > c$ to show the desire metric satisfies the triangle inequality.

4. B. Euler characteristic of genus g is $2 - 2g$. Here $g = 0$, so $\xi(P) = F - E + V = 12 - 17 + 7$.

5. E. Closed subsets of compact spaces are compact (open covers union the complement). Need to show that f is open. Can show it's closed (not true in general, but since f is bijective we can do this). See the website:

   `http://www.proofwiki.org/wiki/Continuous_Bijection_from_Compact`

   `_to_Hausdorff_is_Homeomorphism`

   It's a great exercise to make sure you understand the proof of this. It uses fundamental definitions from point-set topology. It's a rite of passage.

6. B. If it has all it's limit points, then it is $\mathbb{R}$ as $\mathbb{Q}$ is dense.

7. C. Again, all they want you to know is that continuous maps take compact sets to compact sets.

8. B. $X = \cup_{n\in\mathbb{N}}[0, 1-1/n)\bigcup\{1\} = [0, 1/2) \cup [1/2, 2]$

9. E. Let's show why I fails. If $f : \mathbb{R} \to \mathbb{R}$ is given by $x \mapsto x^2$ and $A = [0, 1]$, then $f^{-1}(A^0) = f^{-1}((0, 1)) = (-1, 0) \cup (0, 1)$, whereas $(f^{-1}(A))^0 = (-1, 1)$. For II, let $f : [-1, 2]$ be given by $f(x) = x^2$. When $A = (1, 4)$, then $f^{-1}(\overline{A}) = \{1\} \cup [1, 2]$, wherease $\overline{f^{-1}(A)} = [1, 2]$. III is false because it holds for ALL functions, not just continuous ones.

Combinatorics

1. E. Each player must wina different number of games.

2. D. Direct pigeon hole application for I. For II, place all points in one region to see fallacy. For III. Suppose n is the number of balls in one sector, then $8 - n$ are the balls in adjacent sectors and the remaining balls in the two adjacent sectors not accounted for is $21 - 2(8 - n) - n = 5 + n$. If we choose $n \geq 5$ (and obviously smaller than 9), then the remaining two places can have 9 points.

3. B. There are two ways to do this. The first is to directly count by noticing that we have to have an even number of odds chosen:

$$2^4(\begin{pmatrix} 5 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \end{pmatrix}) - 1$$

   where we have omitted the emptyset. Alternatively, we can show there is a bijection between sets that sum to even and odd numbers simply by mapping a set to another by removing or adding 1 to the set depending on whether 1 is already a member. This shows the bijection, but since this maps $\{1\} \to \emptyset$.

4. C. Binomial theorem! The sum is $(4 - 1)^n$

5. C. It is a string of 12 letters (7 N's and and 5 E's).

6. $\begin{pmatrix} 12 \\ 5 \end{pmatrix} - \begin{pmatrix} 8 \\ 3 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

7. B. You need to write them all down. We are assuming the trees are unlabeled and equivalent with respect to graph isomorphism.

8. E. Whatever Alan plays, Barbara mirrors his moves on a column (say we have paired off columns at the beginning).

9. E. First choose k spots in the domain and then order up the way k points in the domain can match them.

10. E. There are 3 possible remainders $0, 1, 2$. There are $\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6$ possible differences. By the pigeon hole principle, there always is at least one n.

Probability

1. D. It's a geometric series that looks like: $1/6 + (5/6)^2(1/6) + (5/6)^4(1/6) + \ldots$
2. B. $1 - (1/2 + 1/4 + 1/8)^2$
3. D. Obviously between C and D. Note that D can be written as $P(48 \leq H \leq 52)$ and hence is more concentrated about the mean.
4. D. Get mean: $\int_{-1}^{1} \frac{3}{4}(x - x^3) = 0$ and hence the variance is:

$$\int_{-1}^{1} \frac{3}{4}(x^2 - x^4) = (3/2)[(1/3) - 1/5) = 1/5$$

5. D. It's the region in a 1 by 1 square in the xy-plane such that $|x - y| < 1/2$.
6. B. This is a geometric distribution so $1/(1/100)$ where $1/100$ is the probability of termination/ending the process. To derive easily: $X =$ length of the game and $E(X)$ is the expectation, then $E(X)$ satisfies: $E(X) = p(1) + (1-p)(E(X))$ where p is the probability of termination/ending the process (in our case $p = 1/100$ as said); then solve for $E(X)$ to get $1/p$
7. B. Similarly, we see that $E(X)$ satisfies:

$$E(X) = 1/4(2) + 1/2(E(X) + 1) + 1/4(E(X) + 2)$$

and then solve for $E(X)$. The first term in the sum is when we roll two heads. The second term in the sum is when we first flip a heads, and the last term represents the scenario when we flip a heads and then a tails.

Complex Analysis

1. A. Polynomials are holomorphic.
2. C. This is standard for roots of unity question. Note that if $\{z | z^n - 1\}$, then this is a group under multiplication and it is cyclic generated by $e^{2\pi i/n} = g$. Easily, $\overline{g^k} = g^{-k}$ and all homorphisms are determined where they send g.
3. A. This is just testing that you know that holomorphic is the same as analytic.
4. E. Have to use $1 + x + \ldots + x^{n-1}$ for nth roots of unity not equal to 1.
5. E. Can plug in or can use that Mobius transformations. First note you can map any three points z_1, z_2, z_3 to $0, 1, \infty$ by:

$$f_1(z) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_1}{z_2 - z_3}$$

and then invert (can actually treat coefficients as a matrix and use standard formula)

Euclidean

1. D. We clearly know they intersect twice by a quick sketch. But to get the third point have to know that $x^{2012} < 5^x$ for large enough x.

2. D. Differentiate $1/(1+x)$ and evaluate at $x = 1/2$.

3. E. The dihedral group is literally the group of symmetries of a pentagon, which is clearly the same as this pentagram.

4. E. The key is to connect the radius of the large white circle to the radius of the gray circle. Then, make a right triangle using the tangent hypothesis.

5. B.

$$0 < s - r < 1$$

where we have use the triangle inequality for the latter inequality. Geometrically speaking (and non-rigorously, the difference $s - r$ will be $\sin 20$.

Let's see why using the law of cosines:

$$\sqrt{r^2 + 1 - 2r\cos 110} - r = \sqrt{(r - \cos 110)^2 + k} - r$$

where k is some unspecified constant. As $r \to infty$ the limit is clearly $-\cos 110$ (use the standard trick of multiplying by the conjugate, i.e. $\sqrt{(r - \cos 110)^2 + k} + r / \sqrt{r - \cos 110)^2 + k} + r$, etc.

Multivariable Calc

1. C. Note this is but $\arctan x + \arctan y$! Then evaluate $\frac{\partial z}{\partial x}$ at $x = 2$ to get: $1/5$

2. B. The integral is $\int_0^1 \int_0^{\sqrt{1-x^2}} 8xydydx$ and integrate.

3. C. We need to evaluate:

$$\int_0^1 \vec{F} \cdot d\vec{s} = \int_0^1 (-1, 0, 1) \cdot \dot{\gamma} dt = \int_0^1 -1 + 3t^2 dt = 0$$

4. C. Note this is a plane intersecting the solid sphere and hence must be maximized or minimized on the boundary. We then translate this into a lagrange multiplier porblem:

$$\begin{cases} \lambda = 2x \\ 0 = 2y \\ 4\lambda = 2z \end{cases}$$

Doing algebra gives us $x = \pm\sqrt{34}/17$ and $z = \pm 4\sqrt{17}/17$. Plugging into the original function gives us the result.

5. A. Change to polar coordinates and get $\int_{\pi/2}^{\pi}\int_0^{2a\sin\theta} r(rdrd\theta) = \int_{\pi/2}^{\pi} \frac{8}{3}a^3\sin^3\theta d\theta = \frac{16}{9}a^3$. Note that we are integrating over the circle to the left of the y-axis with center $(0, a)$, equation $x^2 + (y-a)^2 = a^2$, and radius a. We easily see that $\sqrt{2ay - y^2} = \sqrt{a^2 - (y-a)^2}$.

D.E.

1. E. It's separable and integrating once we obtain:

$$\frac{1}{y'} = C_1 - x$$

Integrating once more we obtain:

$$C_2 - \ln|C_1 - x| = y$$

But if we let $C_1 = -C_1'$, then $C_2 - \ln|C_1 + x|$ or the same as answer E, without the absolute values, which just means we have restricted the domain.

2. C. It's almost exact. Notice that this homogeneous of degree 1. The trick is to set $y = xv$ as a substitution for y. Hence, $dy = vdx + xdv$ and $v = y/x$. In particular,

$$x^2dv + (2xv - xe^x)dx$$

which is now exact! Hence, our solution is: $x^2v - xe^x + e^x = C \rightsquigarrow xy - xe^x + e^x = C$. Substituting $(1, 0)$ gives us $C = 0$. Hence, we $2y - e^2 = 0$ and so $e^2/2$.

3. B. Integration factor is x and then use integration by parts to get:

$$-\cos(x) + \frac{1}{x}\cdot\sin x + C = y$$

Then we see that $C = 0$ from the initial condition and so $f(\pi/2) = \frac{2}{\pi}$

4. A. We can rewrite this D.E. as:

$$y' - y/x = 1$$

The integration factor $1/x$ and so we get:

$$x\ln x + C = y$$

A slight reqrrangement gives the answer.

5. E. This is linear D.E. however the right hand side is the solution to the homogenous one so we must guess a particular solution of the form At^2e^t.